

# Global knot theory in $F^2 \times \mathbb{R}$

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## Abstract

We introduce a special class of knots, called global knots, in  $F^2 \times \mathbb{R}$  and we construct new isotopy invariants, called  $T$ -invariants, for global knots.

Some  $T$ -invariants are of finite type but they cannot be extracted from the generalized Kontsevitch integral (which is consequently not the universal invariant of finite type for the restricted class of global knots).

We prove that  $T$ -invariants separate all global knots of a certain type. As a corollary, we prove the non-invertibility of some links in  $S^3$  without making use of the link group.

## 1 Introduction and main results

Let  $F^2$  be a compact oriented surface with or without boundary, and let  $v$  be a Morse-Smale vector field on  $F^2$  which is transversal to the boundary  $\partial F^2$ . (For us, a Morse-Smale vector field is a smooth vector field, having at most isolated non-degenerated singularities, and no limit cycles.) We study oriented knots  $K$  in the oriented manifold  $F^2 \times \mathbb{R}$ .

It turns out that there are naturally three types of knot theory, which we call respectively local, global and general.

### 1.1 Local knot theory

Let  $F^2$  be the disk  $D^2$  or the sphere  $S^2$  and let  $v$  be a vector field which has only critical points of index +1. Alexander's theorem says that each knot type (i.e. a knot up to ambient isotopy) has a representative, also called  $K$ , such that the projection  $K \hookrightarrow F^2 \times \mathbb{R} \rightarrow F^2$  is transversal to  $v$ . Markov's theorem says that two transversal representatives of the same knot type can be joined by an almost transversal isotopy i.e. an isotopy through transversal knots, besides for a finite number of values of the parameter where a Reidemeister move of type I occurs in a singularity of  $v$  (such a move is usually called Markov move). This type of knot theory is traditionally the most studied one and we cannot add anything new here.

### 1.2 Global knot theory

A knot type  $K$  is called a *global knot* if there is a vector field  $v$  without critical points or only with critical points of index -1, and there is a representative  $K$  such that the projection  $K \hookrightarrow F^2 \times \mathbb{R} \rightarrow F^2$  is transversal to  $v$ . We call such  $v$  *non-elliptic* vector fields. Notice that a knot  $K \hookrightarrow D^2 \times \mathbb{R} \hookrightarrow F^2 \times \mathbb{R}$  can never be a global knot. This implies that there is no analogue of Alexander's theorem here. However, there is an analogue of Markov's theorem for global knots which is even better than Markov's theorem: *If two representatives of the same global knot have projections transversal to the same non-elliptic vector field  $v$ , then there is an isotopy between them transversal to  $v$ .* We give a proof in a special case and we indicate the idea of the proof in the general case.

Global knots are the main object of our work. They have almost never been studied before, except for the very special case of closed braids.

### 1.3 General knot theory

A knot type  $K$  is called a *general knot* if for each representative  $K$  the following property is verified: if  $v$  is a vector field such that  $K \hookrightarrow F^2 \times \mathbb{R} \rightarrow F^2$  is transversal to  $v$ , then  $v$  has critical points of both indices  $+1$  and  $-1$ . General knots do exist. For example: the Whitehead link seen as a knot in the solid torus. Evidently, if a knot is neither local nor global, then it is general. We prove that in the general setting there is no analogue of Markov's theorem (in difference to the global setting). More precisely, we give two representatives of some general knot in the solid torus with projections transversal to the same vector field  $v$ , and we show that they cannot be joined by any transversal isotopy. This indicates that the general case is even much more complicated than the local and the global case.

The main achievement of our work is the construction of new isotopy invariants, called *T-invariants* ( $T$  means "transversal"), for global knots. These invariants depend neither on the chosen non-elliptic vector field  $v$ , nor on the chosen representative of the knot whose projection is transversal to  $v$ . Hence,  $T$ -invariants are isotopy invariants in the usual sense.  $T$ -invariants are defined as "Gauss diagram invariants". Consequently, their calculation has polynomial complexity with respect to the number of crossings of the knot diagrams. However, not all  $T$ -invariants are of finite type in the sense of Vassiliev. Moreover, we show that even some  $T$ -invariants of finite type cannot be extracted from the generalized Kontsevitch integral (see [A-M-R]). This comes from the fact that  $T$ -invariants are well defined only for global knots and not for all knots in  $F^2 \times \mathbb{R}$ .

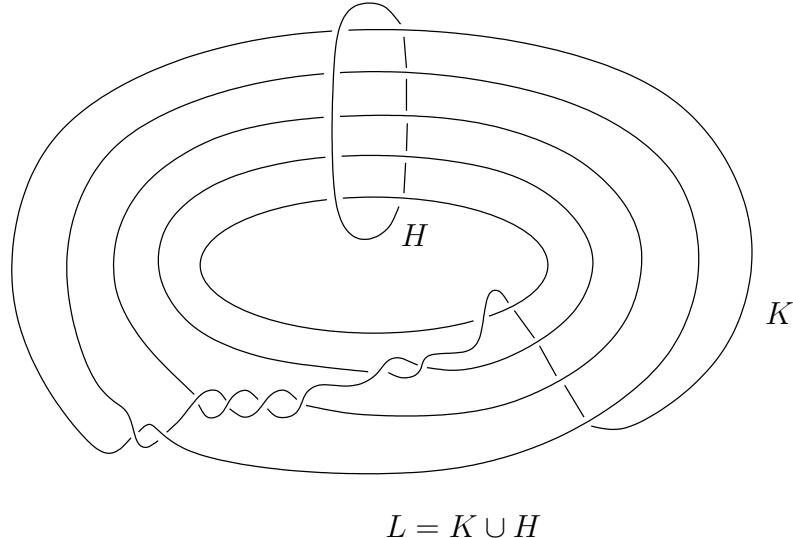
A knot  $K \hookrightarrow F^2 \times \mathbb{R}$  is called a *solid torus knot* (or a closed braid) if it has a representative whose projection is contained in some annulus  $S^1 \times I \hookrightarrow F^2$  (of course, local knots are a special case of solid torus knots). We conjecture that  $T$ -invariants separate all global knots in  $F^2 \times \mathbb{R}$  which are not solid torus knots.

We prove this conjecture in the following special case: Let  $T^2$  be the torus. An oriented global knot  $K \hookrightarrow T^2 \times \mathbb{R}$  is called  $\mathbb{Z}/2\mathbb{Z}$ -pure if for each crossing of  $K$ , each of the two oriented loops obtained by splitting the crossing is non-trivial in  $H_1(T^2; \mathbb{Z})/\langle [K] \rangle \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$  (We consider  $K$  as a diagram in  $F^2 \times \mathbb{R}$  over  $F^2$  and we denote by  $[K]$  the homology class represented by  $K$ . If  $K$  is not a solid torus knot then  $H_1(T^2; \mathbb{Z})/\langle [K] \rangle \otimes \mathbb{Z}/2\mathbb{Z}$  is automatically isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .)

**Theorem 1.1**  *$T$ -invariants separate all  $\mathbb{Z}/2\mathbb{Z}$ -pure global knots in  $T^2 \times \mathbb{R}$ .*

Let  $flip : T^2 \times \mathbb{R} \rightarrow T^2 \times \mathbb{R}$  be the hyper-elliptic involution on  $T^2$  multiplied by the identity on the lines  $\mathbb{R}$ . An oriented knot  $K \hookrightarrow T^2 \times \mathbb{R}$  is called *invertible* if it is ambient isotopic to  $flip(-K) = -flip(K)$ . One easily shows that both the generalized HOMFLY-PT and the generalized Kauffman polynomial can never distinguish  $K$  from  $flip(-K)$ . (The polynomials of the cables do not make the distinction either.) On the other hand, we prove the non-invertibility of some global knots in  $T^2 \times \mathbb{R}$  using  $T$ -invariants of finite type. Consequently, these invariants cannot be extracted from the above knot polynomials.

Knots in  $T^2 \times \mathbb{R}$  are in 1-1 correspondence with ordered 3-component links in  $S^3$  containing the Hopf link  $H$  as a sublink. Using a  $T$ -invariant of degree 6 for the knot  $K$ , we show that the link  $L = K \cup H \hookrightarrow S^3$  (see Fig. 1). is not invertible for any chosen orientation on it. Notice that this is the first proof of the non-invertibility of a link with a numerical invariant, which does not make any use of the link group  $\pi_1(S^3 \setminus K; *)$ . (Compare with  $L$  for approaches which use the link group.)



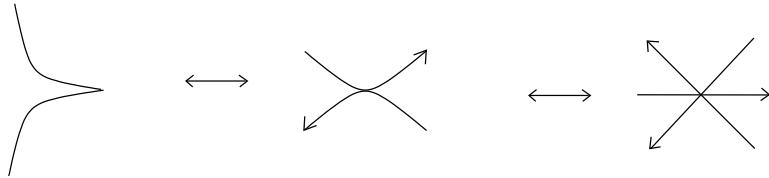
**Fig. 1**

Let us consider the space of all diagrams of a given knot type  $K \hookrightarrow F^2 \times \mathbb{R}$ . The discriminant is the subspace of all non-generic diagrams. Each

path in the space of diagrams which cuts the discriminant only in strata of codimension 1 is a generic isotopy of knots (see [F], sect. 1).

The construction of  $T$ -invariants relies on the combination of two different approaches. On one hand, there is the concept of  $G$ -pure knots and  $G$ -pure isotopy. A  $G$ -pure isotopy is an isotopy which does not cut the discriminant in strata with certain homological markings of the crossings.  $T$ -invariants are, roughly speaking, Gauss diagram formulas which are invariant under  $G$ -pure isotopy. The problem is now, that even if two knots are isotopic, we cannot grant that there exists a  $G$ -pure isotopy joining them.

At this place, the concept of global knots is introduced. If two isotopic knots are global, then there exists an isotopy through global knots between them. Such an isotopy does not cut the strata of the discriminant depicted in Fig. 2.



**Fig. 2**

We still do not know whether isotopic  $G$ -pure global knots can be joined by a  $G$ -pure isotopy. But for an isotopy through global knots, we have enough control over the "cycles of crossings" (compare with [F], sect. 4) to enable us to show that  $T$ -invariants are actually isotopy invariants!

Using  $T$ -invariants, we show in sect. 9 that there exist isotopic  $G$ -pure general knots such that

1. there is no  $G$ -pure isotopy between them
2. there is no isotopy transversal to the vector field  $v$  between them

## 2 Isotopy of global knots

We fix orientations on  $F^2$  and  $F^2 \times \mathbb{R}$ . Let  $pr : F^2 \times \mathbb{R} \rightarrow F^2$  be the canonical projection, and let  $v$  be a non-elliptic vector field. We fix the orientation on the global knot  $K \hookrightarrow F^2 \times \mathbb{R}$  in such a way that  $(pr(K), v)$  induce the given orientation on  $F^2$ .

The natural equivalence relation for global knots is: ambient isotopy through global knots keeping the vector field  $v$  fixed. However, we believe that this equivalence relation coincides with the usual ambient isotopy.

*Conjecture.* *Two global knots with respect to the same vector field  $v$  are ambient isotopic if and only if they are ambient isotopic through global knots with respect to  $v$ .*

**Remarks:** 1. The conjecture is true in the case of closed braids: this is a consequence of Artin's theorem.

2. Below we give a proof in the case where  $F^2 = T^2$ .
3. We outline the strategy of the proof in the general case (and we will come back to it in another paper): One can easily prove in a geometrical way that braids are isotopic as tangles if and only if they are isotopic as braids (see e.g. [P-S]). This proof can be generalized to all such isotopies of global knots which do not pass through the singularities of  $v$ . If an isotopy passes through a singularity of  $v$ , then the resulting knot is no longer transversal to  $v$ , except for the case where the isotopy passes again through the same singularity but in the opposite direction. This can be proven by constructing 2-disks with piecewise smooth boundary in the trace of the isotopy on  $F^2$ . The vector field  $v$ , having only critical points of index -1, can never be transversal to such a disk. Using this fact, one can remove successively these 2-disks in the isotopy.

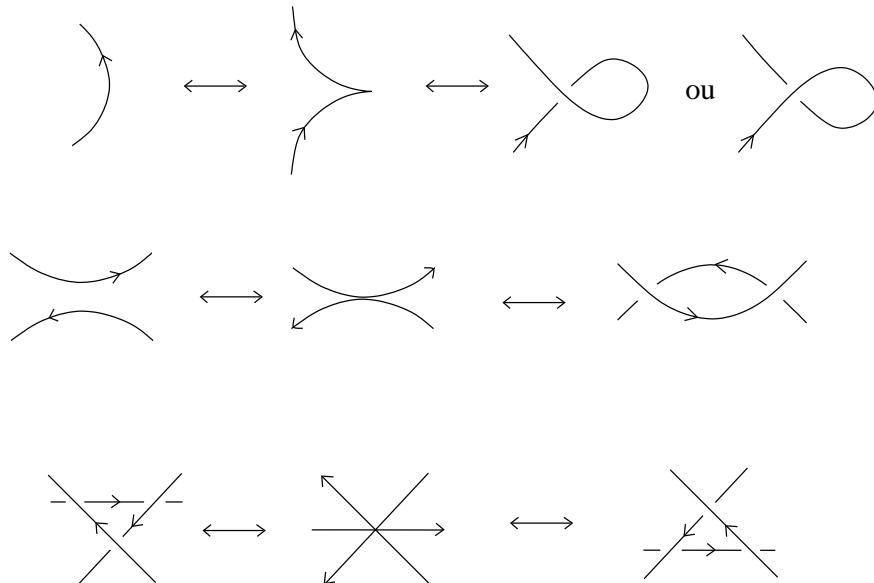
We consider now the case of the torus  $T^2$ . Let  $pr_2 : S^1_1 \times S^1_2 \rightarrow S^1_2$  be the projection and let  $v$  be the unit vector field tangential to the fibres of  $pr_2$  (we might perturbate  $v$  slightly so that it has no longer any closed orbits). Consequently, a knot  $K \hookrightarrow T^2 \times \mathbb{R}$  is global with respect to  $v$  if and only if the restriction  $pr_2 : K \rightarrow S^1_2$  is a covering.

**Lemma 2.1** *Isotopic global knots with respect to  $v$  in  $T^2 \times \mathbb{R}$  are isotopic through global knots with respect to  $v$ .*

Artin's theorem implies that isotopic closed braids are isotopic through closed braids (see e.g. [M]). Let  $T^2 \times \mathbb{R} \hookrightarrow S^3$  be the standard embedding and let  $A_1$  resp.  $A_2$  be the cores of the solid tori  $S^3 \setminus (T^2 \times \mathbb{R})$ . The oriented link  $A_1 \cup A_2$  is determined by the following conventions:  $lk(A_1, \{*\} \times S^1_2) = lk(A_2, S^1_1 \times \{*\}) = 1$  and  $K$  is a global knot in  $T^2 \times \mathbb{R}$  with respect to  $v$  if and only if  $K \cup A_2$  is a closed braid for the natural fibering  $S^3 \setminus A_1 \rightarrow S^1_2$ . Consequently, if  $K$  and  $K'$  are isotopic global knots then  $K \cup A_2$  and  $K' \cup A_2$  are closed braids which are isotopic as links. According to Artin's theorem, they are also isotopic as closed braids. But  $A_2$  is just the closure of a 1-string braid and therefore we may assume that it remains fixed in the isotopy of closed braids. This implies that the isotopy of  $K$  in  $S^3 \setminus (A_1 \cup A_2)$  is an isotopy through global knots ■.

From now on, all the isotopies considered will be isotopies through global knots.

*Basic observation.* *In an isotopy of global knots, the Reidemeister moves depicted in Fig. 3 can never occur.*



**Fig. 3**

**Proof.** Obviously, in the singularities there is always one branch of  $K$  which has not the correct orientation (see the middle part of Fig. 3). ■

**Question.** *Is there an analogue of Alexander's theorem (in the right sense) for knots in  $T^2 \times \mathbb{R}$ ? More precisely, given a diagram  $K$ , is there*

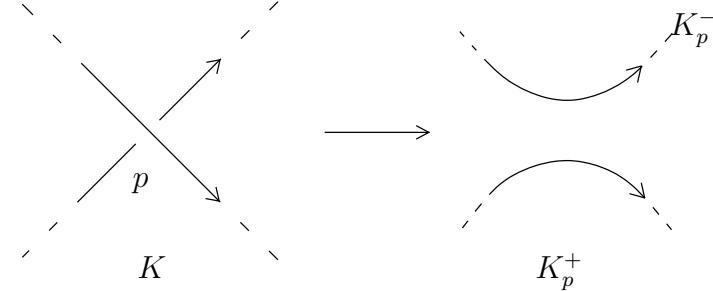
an algorithm which either constructs some non-elliptic vector field  $v$  and a representative of  $K$  transversal to  $v$ , or otherwise, which proves that  $K$  is not a global knot?

### 3 Construction of $T$ -invariants for global knots.

Let  $K_0 \hookrightarrow F^2 \times \mathbb{R}$  be an oriented global knot and let  $K_t, t \in [0, 1]$  be an isotopy of  $K_0$  (through global knots). Let  $G$  be a fixed quotient group of  $H_1(F^2; \mathbb{Z})$  and let  $[K]_G$  be the homology class in  $G$  represented by  $K = K_0$ .

**Definition 3.1** A global knot  $K$  is  $G$ -pure if for each crossing  $p$  of  $K$  (with respect to the projection  $pr$ ),  $[K_p^+]_G \notin \{0, \pm[K]_G\}$ . The isotopy  $K_t$  is  $G$ -pure if  $K_t$  is  $G$ -pure for each  $t$ . A knot type (called  $K$  as the knot itself) is  $G$ -pure if it has a global representative which is  $G$ -pure.

We remind the definition of the oriented (global) knot  $K_p^+$  in Fig. 4 (see also [F], sect. 0 and 1).



**Fig. 4**

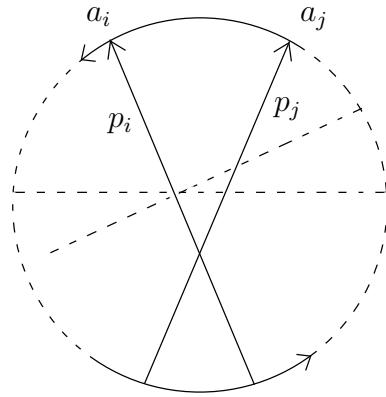
The *Gauss diagram* of  $K$  is the abstract chord diagram of  $pr(K)$ , where each chord  $p$  is oriented from the undercross to the overcross of the crossing  $p$ , and it is marked by the writhe  $w(p)$  and by the homology class  $[K_p^+]_G \in G$ . A *configuration* is a given (abstract) chord diagram with given orientations and homological markings (in  $G$ ) of the chords, but without writhes. A *Gauss sum of degree  $d$  for the knot  $K$*  (also called a *Gauss diagram formula*) is a sum which is defined in the following way: Let  $D$  be a given configuration of  $d$  chords. We consider the integer

$$\sum_D \text{function (writhes of the crossings of } K \text{ corresponding to the chords of } D)$$

where  $D$  runs over all the subdiagrams of the Gauss diagram of  $K$ . The function is called the *weight function* (see also [F], sect. 0 and 1).

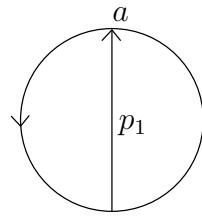
**Definition 3.2** A  $G$ -pure configuration of degree  $m$  is a given configuration  $D$  of  $m$  oriented chords  $(p_1, \dots, p_m)$  with corresponding markings  $(a_1, \dots, a_m)$  where each  $a_i \in G \setminus \{0, \pm[K]_G\}$ , verifying the following 2 conditions:

1. Let  $D$  be represented as a subdiagram  $D_0$  of a Gauss diagram of any  $G$ -pure knot  $K_0$ , and let  $K_t, t \in [0, 1]$  be any  $G$ -pure isotopy of  $K_0$  without Reidemeister moves of type  $II$  involving one of the crossings  $p_i$ . Then,  $D$  is preserved as a subdiagram of  $K_t$  i.e. there is a continuous family  $D_t$  of subdiagrams of the Gauss diagrams of  $K_t$  and almost each  $D_t$  represents  $D$ .
2. If  $D$  contains a fragment of the type depicted in Fig. 5, then  $a_i \neq a_j$ .



**Fig. 5**

*Exemple.* 2.1  $G := \mathbb{Z}$  and  $[K]_G = 0$ . Let  $a \in \mathbb{Z} \setminus 0$  be fixed. There is exactly one  $\mathbb{Z}$ -pure configuration of degree 1 which involves the class  $a$  (see Fig. 6). There are exactly three  $\mathbb{Z}$ -pure configurations of degree 2 which involve the class  $a$  (see Fig. 7).



**Fig. 6**

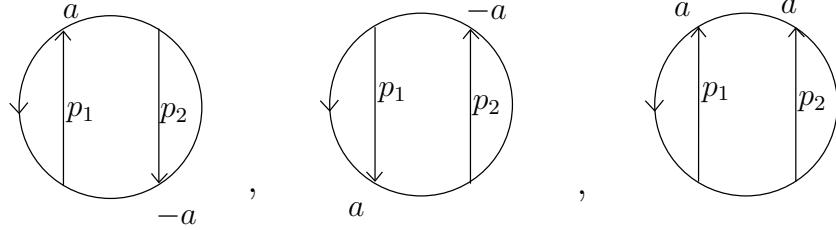


Fig. 7

This follows immediately from the fact that a  $G$ -pure isotopy does not intersect the following strata in the discriminant:  $a_{\circlearrowleft}^{+(-)}(0|a, 0)$ ,  $a_{\circlearrowleft}^{+(-)}(0|a, -a)$ ,  $a_{\circlearrowleft}^{+(-)}(a|a, 0)$ ,  $a_{\circlearrowleft}^{+(-)}(a|0, a)$ , (See [F], sect. 4.11).

In fact, in [F] we replaced all triple points of type by triple points of type . A closer look to this replacement shows that a Reidemeister move of type  $III$  corresponding to is  $G$ -pure and a configuration  $D$  is invariant under this move if and only if it is invariant under the corresponding move of type . We illustrate this in Fig. 8.

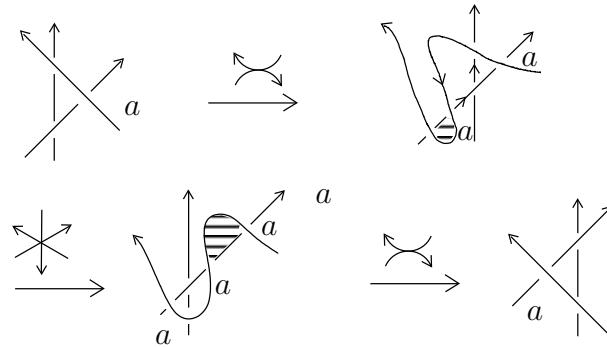


Fig. 8

The homological markings involved in are exactly the same as those involved in the corresponding . Notice that the  $G$ -pure configuration depicted in the left-hand part of Fig. 9, which appears in the replacement, disappears again.

Consequently, the chords  $p_1$  and  $p_2$  of the above configurations cannot become crossed (as shown in the right-hand part of Fig. 9) in a  $G$ -pure isotopy.



Fig.9

*Example 3.2*  $G := \mathbb{Z}/2\mathbb{Z}$  and  $[K]_G = 0$ . Each chord is marked by the non-trivial element in  $\mathbb{Z}/2\mathbb{Z}$ . In this case, each configuration which does not contain a fragment as depicted in Fig. 10

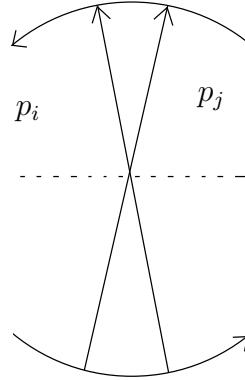


Fig. 10

is a  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration.

Let  $D$  be a fixed  $G$ -pure configuration of degree  $m$ .

**Definition 3.3** Let  $\mathcal{D}_i, i \in I$  be a finite set of configurations, each of them with at most  $m + n$  oriented chords  $(p_1, \dots, p_m, q_1^{(i)}, \dots, q_{n_i}^{(i)})$ . Let  $n = \max_i(n_i)$ . All chords have markings in  $G \setminus \{0, \pm[K]_G\}$ . The given (unordered) chords  $(p_1, \dots, p_m)$  form the given  $G$ -pure subconfiguration  $D$  in each  $\mathcal{D}_i$ . Let  $f_i, i \in I$  be functions

$$f_i : \underbrace{\mathbb{Z}/2\mathbb{Z} \times \dots \times \mathbb{Z}/2\mathbb{Z}}_{n_i} \rightarrow \mathbb{Z}$$

The linear combination of Gauss diagram formulas

$$c(D) := \sum_{\mathcal{D}_i} f_i(w(q_1^{(i)}), \dots, w(q_{n_i}^{(i)}))$$

is called a *G-pure class of  $D$  of degree at most  $n$*  if the following condition holds:

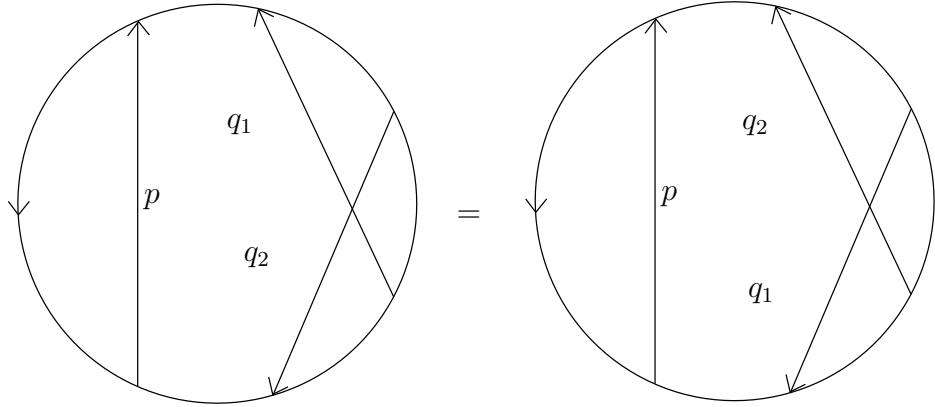
Let  $D$  be represented as a subdiagram  $D_0$  of a Gauss diagram of any  $G$ -pure knot  $K_0$  and let  $K_t, t \in [0, 1]$  be any  $G$ -pure isotopy of  $K_0$  without Reidemeister moves of type  $II$  involving one of the  $p_i$ . (Hence, there exists a continuous family  $D_t$  as in Def. 3.2.). Let

$$c(D_t) := \sum_i \sum_{\Delta_i} f_i(w(q^{(i)}_1, \dots, w(q^{(i)}_{n_i})))$$

where  $\Delta_i$  runs through all subdiagrams which represent  $\mathcal{D}_i$  in the Gauss diagram of  $K_t$ , and which contain  $D_t$  as the given subconfiguration  $D$ . The integer  $c(D_t)$  is the same for all  $t \in [0, 1]$  such that the projection  $pr : K_t \rightarrow F^2$  is generic. (Here,  $w(q_j^{(i)})$  is the writhe of the crossing  $q_j^{(i)}$ .)

**Remarks:**

1. In the calculation of  $c(D_t)$ , subdiagrams which coincide up to different numerations of the chords are identified,

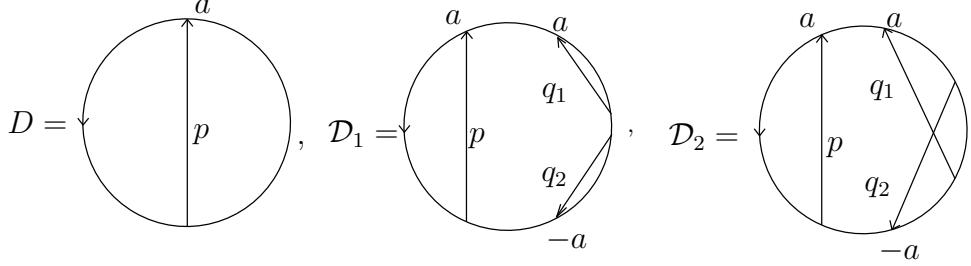


**Fig. 11**

so that they bring only one term into the sum (see Fig 11).

2. We say that the class  $c(D)$  is of degree  $n$  if for each class  $c'(D)$  which uses no more than  $n - 1$  chords,  $c(D_t) - c'(D_t)$  is not constant.

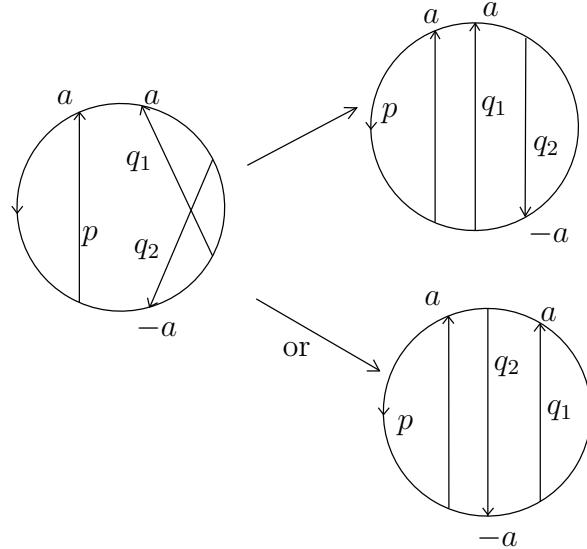
*Example 3.3*  $G := \mathbb{Z}$ ,  $[K]_G = 0$ ,  $a \in \mathbb{Z} \setminus 0$ ,  $D$ ,  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  are as in Fig. 12,  $f_1 \equiv f_2 = w(q_1)w(q_2)$ .



**Fig. 12**

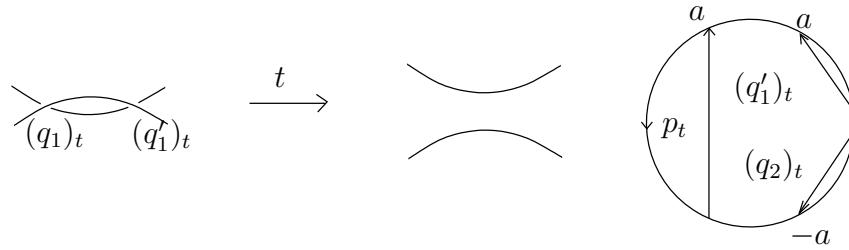
$$c(D) = \sum_{\mathcal{D}_1} w(q_1)w(q_2) + \sum_{\mathcal{D}_2} w(q_1)w(q_2)$$

is a class of  $D$  of degree 2. Indeed, in any  $G$ -pure isotopy of a knot  $K$  such that the crossing  $p$  does not disappear, we observe the following: the chord  $p$  can get crossed with none of the  $q_i$ ,  $i = 1, 2$ . The chords  $q_1$  and  $q_2$  can get crossed together by passing e.g. a stratum of the type  $a^+(a| -a, 2a)$ . But we count them now in  $\mathcal{D}_2$  instead of  $\mathcal{D}_1$ . Notice that the move depicted in Fig. 13 is again not possible.



**Fig. 13**

Let e.g.  $(\mathcal{D}_1)_t$  be a subdiagram which represents  $\mathcal{D}_1$  for a knot  $K_t$ . If in the isotopy e.g.  $(q_1)_t$  disappears (see the left-hand part of Fig. 14), then there is a crossing  $(q'_1)_t$  such that the diagram depicted in the right-hand part of Fig. 14 represents also  $\mathcal{D}_1$ . But for the writhes  $w((q_1)_t) = -w((q'_1)_t)$ , and hence for  $f_1 = f_2 = w(q_1)w(q_2)$ , the contributions of  $(q_1)_t$  and  $(q'_1)_t$  in  $c(D_t)$  cancel out. This shows that  $c(D)$  is a class, and calculating examples one easily establishes that it is of degree 3.



**Fig. 14**

Let  $K \hookrightarrow F^2 \times \mathbb{R}$  be a  $G$ -pure global knot and let  $D$  be a  $G$ -pure configuration of degree  $m$ . Let  $c_i(D), i \in \{1, \dots, k\}$  be a finite collection of  $G$ -pure classes of  $D$  having degree  $n_i$  respectively, and let  $c_i, i \in \{1, \dots, k\}$  be fixed integers. Let  $n = \max_i n_i$ .

**Definition 3.4** The  $T$ -invariant  $T_K(D; c_1(D) = c_1, \dots, c_k(D) = c_k)$  for  $G$ -pure global knots is defined as:

$$\sum_D w(p_1) \cdots w(p_m)$$

Where we sum over all  $D$  occurring as subdiagrams of the Gauss diagram of  $K$ , such that  $c_1(D) = c_1, \dots, c_k(D) = c_k$ . Here,  $w(p_i)$  are the writhes of the crossings of  $K$  which correspond to the chords of the subdiagram representing  $D$ . If  $n = 0$  and  $m \neq 0$ , then  $T_K(D; \emptyset)$  is defined as

$$\sum_D w(p_1) \cdots w(p_m)$$

Where we sum over all  $D \subset$  Gauss diagram of  $K$ . If  $m = 0$  and  $n \neq 0$  then  $T_K(\emptyset; c(\emptyset))$  is defined as

$$c(\emptyset) = \sum_i \sum_{\Delta_i} f_i(w(q_1^{(i)}, \dots, q_{n_i}^{(i)}))$$

Where  $\Delta_i$  runs through all subdiagrams which represent  $\mathcal{D}_i$  in the Gauss diagram of  $K$ . If  $n = m = 0$ , i.e.  $D = \mathcal{D}_i = \emptyset$ , then  $T_K(\emptyset, \emptyset)$  is defined as the *free regular homotopy class* of  $pr(K) \subset F^2$ . (This is the universal invariant of degree 0.)

**Definition 3.5** The set  $\{c_1(D), \dots, c_k(D)\}$  is called a *multi-class* of  $D$ .

**Remark.**

If there is no risk of confusion, we will denote shortly by  $T_K$  the invariant  $T_K(D; c_1(D) = c_1, \dots, c_k(D) = c_k)$ .

We are now ready to formulate our main result.

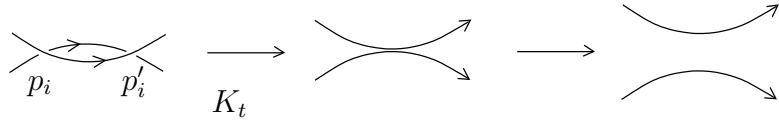
**Theorem 3.1** Let  $K_0$  and  $K_1$  be  $G$ -pure global knots which are ambient isotopic. Then, for each  $T$ -invariant for  $G$ -pure global knots

$$T_{K_0}(D; c_1(D) = c_1, \dots, c_k(D) = c_k) = T_{K_1}(D; c_1(D) = c_1, \dots, c_k(D) = c_k)$$

**Remark.**

The isotopy in the theorem need not to be  $G$ -pure!

**Proof.** The formal proof is very complicated. We just outline the main steps and let the verification of the details for the reader. It follows from the definition of a class  $c(D)$  that  $T_K$  is invariant for all such  $G$ -pure isotopies in which no crossing  $p_i$  disappears for a subdiagram (of the Gauss diagram of  $K$ ) representing  $D$ . Assume now that  $D$  is represented by  $\{p_1, \dots, p_i, \dots, p_m\}$  and that the move depicted in Fig. 15 occurs in the isotopy:



**Fig. 15**

(Remember that  $\curvearrowright$  does not occur in a transversal isotopy.) Then the crossing  $p'_i$ , with which  $p_i$  disappears, verifies:

1.  $w(p_i) = -w(p'_i)$
2.  $p'_i \notin \{p_1, \dots, p_i, \dots, p_m\}$ . (This follows from 2. in Definition 3.2.)
3.  $\{p_1, \dots, p'_i, \dots, p_m\}$  represents  $D$  ( $p_i$  is replaced by  $p'_i$ ).

4.  $c(\{p_1, \dots, p_i, \dots, p_m\}) = c(\{p_1, \dots, p'_i, \dots, p_m\})$ , (in the right-hand term,  $p_i$  is replaced by  $p'_i$ ).

Consequently, the contributions of  $p_i$  and  $p'_i$  in  $T_{K_t}$  cancel out. This implies that  $T_K$  is invariant for  $G$ -pure isotopies.

Assume now that  $K_t, t \in [0, 1]$  is an isotopy which is not necessarily  $G$ -pure (but  $K_0$  and  $K_1$  are  $G$ -pure!). The (value of) a class  $c(D)$  can change only if two of the crossings among  $\{p_1, \dots, p_m; q_1^{(i)}, \dots, q_n^{(i)}\}$  of a configuration  $\mathcal{D}_i$  are involved in a Reidemeister move of type  $III$ , such that the third crossing involved has the homological marking  $0$  or  $\pm[K]_G$ . Performing the isotopy  $K_t$ , let us watch the traces on  $F^2$  of the crossings with markings in  $\{0, [K]_G, -[K]_G\}$ . These traces form immersed circles, called *Rudolph diagrams* (see [F], sect. 4.11). To each such circle, we associate a family of disks  $D_t$  (which are immersed in  $F^2$ ), exactly as in the proof of Theorem. 4.3 in [F]. This is possible because in the isotopy  $K_t$ , there are no Reidemeister moves of type  $II$  with opposite directions of the tangencies. Such moves could destroy the disks  $D_t$ , as shown in Fig. 16.

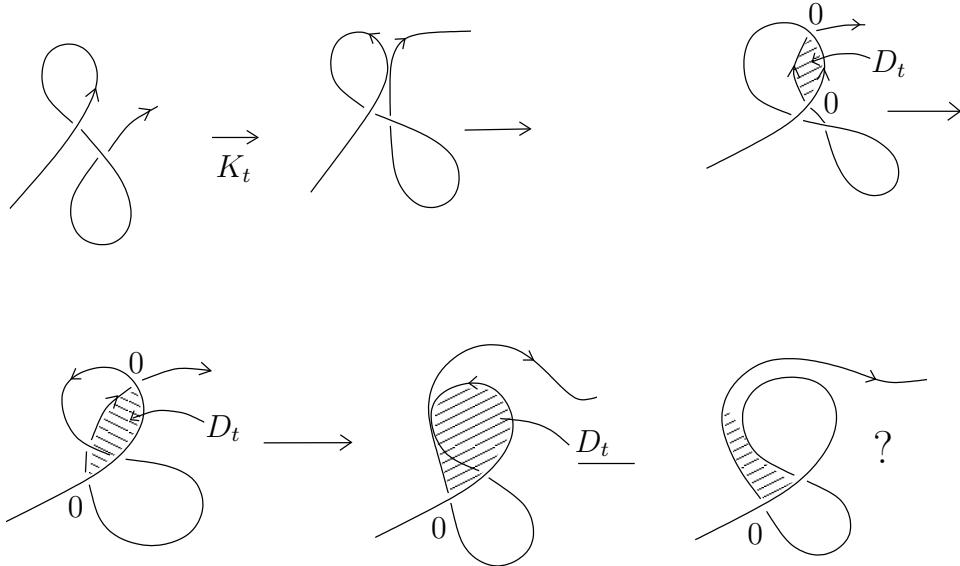


Fig. 16

When Reidemeister moves of type  $II$ , with equal directions of the tangencies, occur in the isotopy  $K_t$ , one gets the usual surgeries of the disks, as shown

in Fig. 17 and 18.

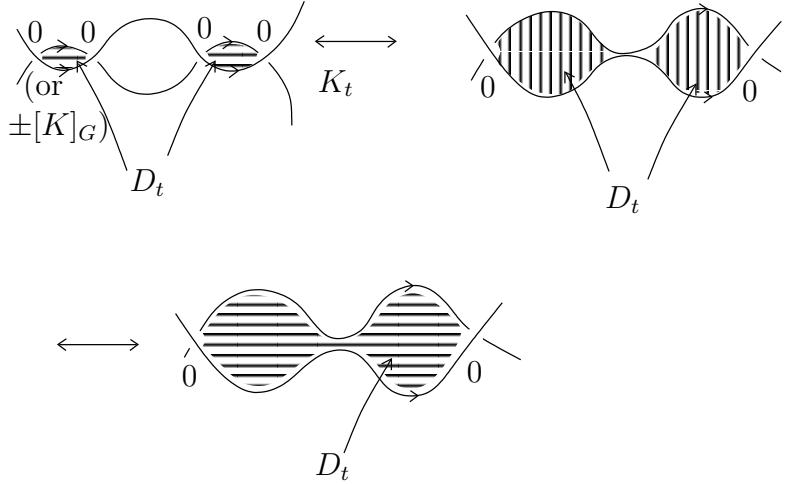


Fig. 17

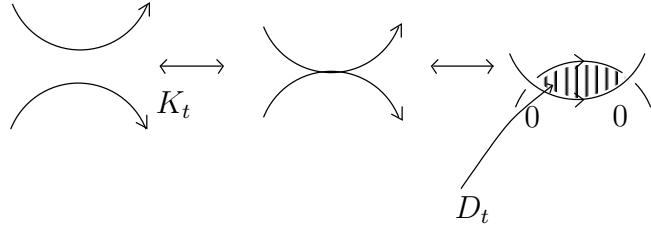


Fig. 18

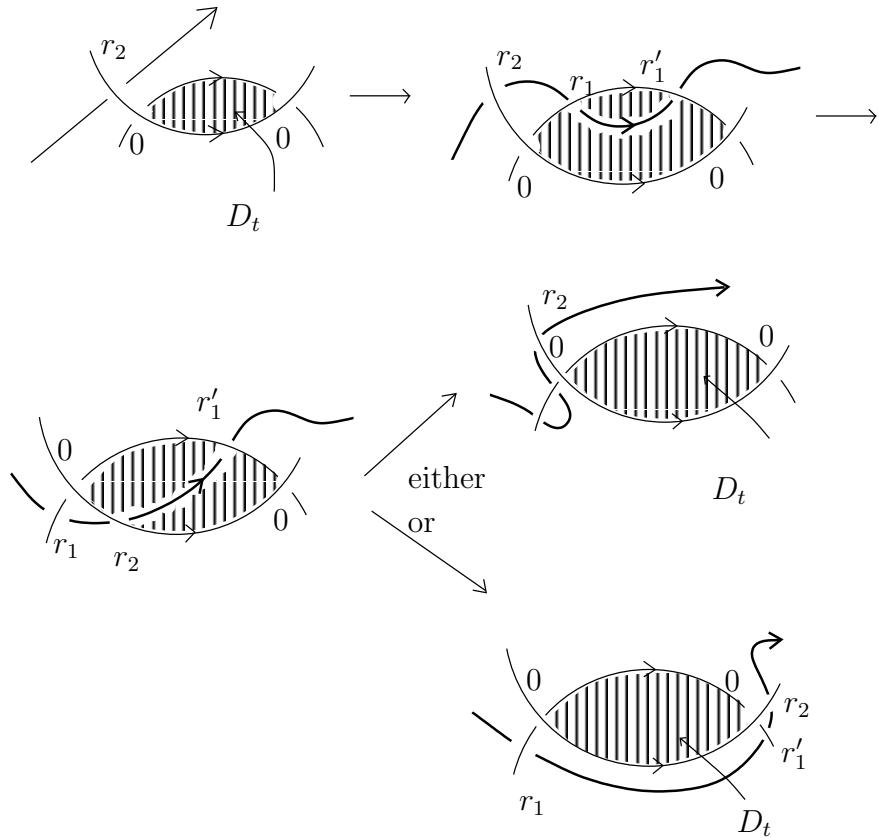
Remember that in a transversal isotopy, there are no Reidemeister moves of type *I*. Each of the families of disks  $D_t$  starts and ends with the empty set because the phenomena explained in Fig. 101 in [F] can still not appear. Each disk in  $D_t$  has exactly two vertices corresponding both to crossings of type 0 or  $[K]_G$  or  $-[K]_G$  which appeared in the underlying circle of Rudolph's diagram.

We distinguish now two cases:

- *The simple case:* Let  $r_1, r_2 \in \{p_1, \dots, p_m; q_1^{(i)}, \dots, q_n^{(i)}\}$  be two crossings of a configuration  $\mathcal{D}_i$  used in the definition of  $T_K$ . We assume that  $r_1$  and  $r_2$  pass together for the first time  $t$  in the isotopy  $K_t$  over a vertex of a disk in  $D_t$ , and neither  $r_1$  nor  $r_2$  passes over a vertex of *another* disk in  $D_t$ .

Then, similar arguments to those used in the proof of Theorem. 4.3 in [F] show that one of the following three possibilities is realized:

1.  $r_1$  and  $r_2$  will pass a second time together over the same vertex, but in opposite directions
2.  $r_1$  and  $r_2$  will pass together also over the second vertex of the disk
3. There exists a third crossing  $r'_1$  which has appeared together with  $r_1$  by a Reidemeister move of type *II* (hence,  $w(r_1) = -w(r'_1)$ , and  $r'_1$  can replace  $r_1$  in the configuration  $\mathcal{D}_i$ ). Moreover,  $r'_1$  will pass together with  $r_2$  over a vertex of the disk (compare with the proof of Theorem. 4.3 in [F]). (See Fig. 19.)



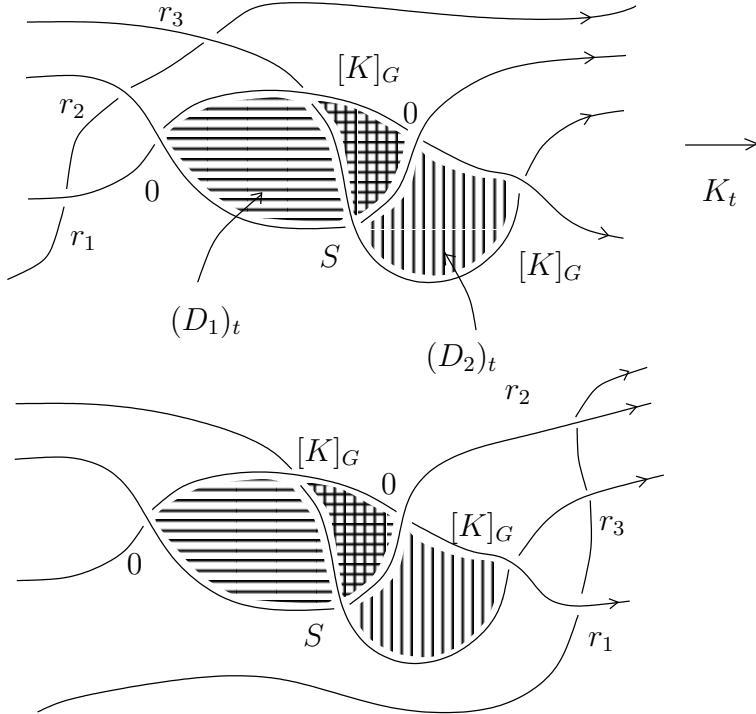
**Fig. 19**

Obviously,  $T_{K_t}$  does not change for the first two possibilities. If in the third possibility,  $r_1$  and  $r_2$  enter together into some configuration

$\mathcal{D}_i$ , then  $r_2$  and  $r'_1$  (instead of  $r_1$ ) enter also in this configuration  $\mathcal{D}_i$ . There are two cases to distinguish:

- A)  $r_1$  and hence  $r'_1$  belong to  $\{p_1, \dots, p_m\} \subset \mathcal{D}_i$ . But then, the contributions of  $r_1$  and  $r'_1$  cancel out in  $T_{K_t}$  because the weight function is  $w(p_1) \cdots w(p_m)$ .
- B)  $r_1$  (and hence  $r'_1$ ) belongs to  $\{q_1^{(i)}, \dots, q_n^{(i)}\} \subset \mathcal{D}_i$ . But then the definition of a class  $c(D)$  implies that the contributions of  $r_1$  and  $r'_1$  cancel out in  $c(D)$ , and hence  $T_{K_t}$  is again invariant.

*The general case:* During the isotopy  $K_t$ ,  $r_1$  and  $r_2$  pass together over a vertex of a disk in  $D_t$ . Let  $t_0$  be the smallest value of  $t$  for which this occurs. At a time  $t_1 > t_0$ ,  $r_1$  passes over the vertex of another disk in  $D_t$ . Notice, that  $r_1$  and  $r_2$  cannot pass together over the vertex of another disk in  $D_t$ . We illustrate the general case with an example, shown in Fig. 20.

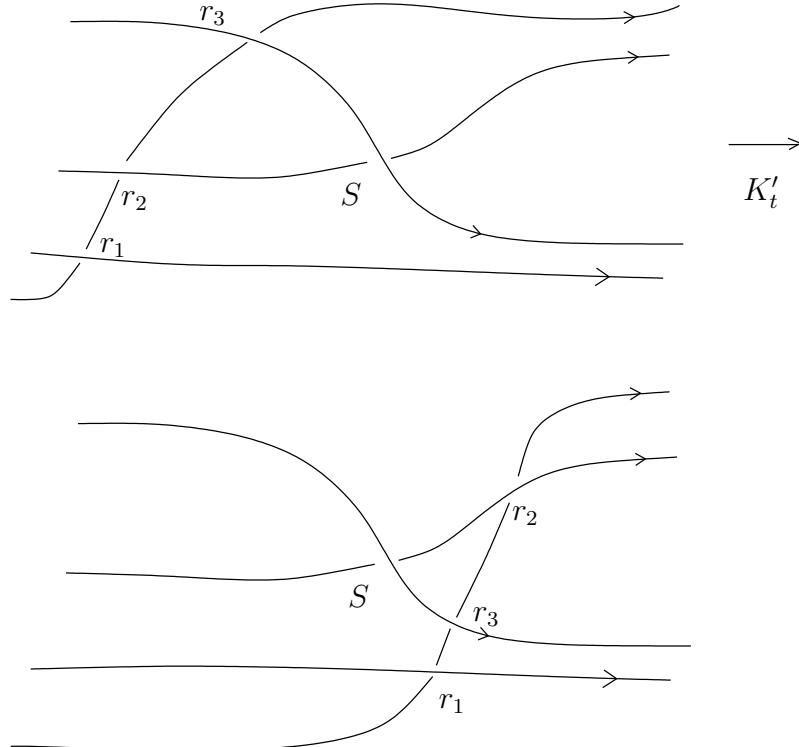


**Fig. 20**

We can assume that the crossing  $s$  in Fig. 20 is not of type 0 or  $\pm[K]_G$  because otherwise, the crossings  $r_2$  and  $r_3$  on the left-hand side would already have passed together over the vertex of a disk in  $D_t$ .

We observe that  $r_1$  with  $r_2$  on the left-hand side of Fig. 20 form exactly the same configurations as  $r_1$  with  $r_3$  on the right-hand side of Fig. 20. But the mutual configuration of  $r_2$  and  $r_3$  has changed (by passing  $s$ ).

Notice that we cannot eliminate the disks  $(D_1)_t$ ,  $(D_2)_t$  by an isotopy, which is supported in a 3-ball containing just the fragment of  $K_t$  drawn in Fig. 20. Nevertheless, we can replace the local piece of the isotopy  $K_t$  shown in Fig. 20 by the local piece of a  $G$ -pure isotopy  $K'_t$  shown in Fig. 21.



**Fig. 21**

The crossings  $r_1, r_2, r_3$  contribute to  $T_{K_t}$  on the left-hand-side (resp. right-hand side) of Fig. 20 exactly the same way as they contribute to  $T_{K'_t}$  in the left-hand side (resp. right-hand side) of Fig. 21. But

the isotopy in Fig. 21 is  $G$ -pure and this implies, as already proven, that  $T_{K'_t}$  is invariant. Consequently,  $T_{K_t}$  for the isotopy  $K_t$  in fig. 20 is invariant too.

Clearly, these arguments can be generalized for the case of more than two disks  $D_t$  and several crossings instead of only  $s$  in the local piece of the isotopy  $K_t$ . ■

**Definition 3.6** Let  $n = \max_i n_i$ . The *degree of  $T_K$  as Gauss diagram invariant* is equal to  $m + n$ .

This definition is justified by the observation that the complexity of the calculation of  $T_K(D; c_1(D) = c_1, \dots, c_k(D) = c_k)$  for knots  $K$  is a polynomial of degree  $m + n$  in the number of crossings of  $K$ . Hence, the invariant  $T_K$  is calculated with the same (order of) complexity as a Vassiliev invariant of degree  $m + n$ .

**Remarks:**

1. Examples 1 and 3 show that the numbers of different  $G$ -pure configurations  $D$  of degree  $m$ , and of classes  $c_i(D)$  of degree  $n$  is in general not finite (the configurations depend on the parameter  $a \in \mathbb{Z} \setminus 0$ ).
2. We show in an example in sect. 9 that multi-classes are usefull. As a matter of fact,  $T_K(D; c_1(D) = c_1, c_2(D) = c_2)$  contains sometimes more information than  $T_K(D; c_1(D) = c_1)$  and  $T_K(D; c_2(D) = c_2)$  together.

**Lemma 3.1** *If  $m = 0$  or  $n = 0$ , then the invariant  $T_K$  is of finite type in the sense of Vassiliev-Gussarov.*

We omit the proof of this lemma: it is a straightforward generalization of Oestlund's proof that the Gauss diagram invariants of Polyak-Viro for knots in  $\mathbb{R}^3$  are of finite type ([P-V], see also [F], sect. 2).

Gussarov has proven that each Vassiliev invariant for knots in  $\mathbb{R}^3$  can be represented as a Gauss diagram invariant of Polyak-Viro (see [G-P-V]). We do not know whether or not this is still true for finite type invariants of global knots. But evidently, each Gauss diagram invariant of finite type which does not use the homological markings  $\{0, \pm [K]_G\}$  in  $G$  is a  $T$ -invariant for  $m = 0$  (i.e.  $D = \emptyset$ ).

**Lemma 3.2** Let  $m > 0$  and  $n > 0$  and assume that  $c(D)$  (from Definition 3.3) is not a Gauss diagram identity, i.e. there exists a knot  $K_t$  and a crossing  $q$  of  $K_t$  such that switching the crossing  $q$  changes the value  $c(D_t)$  (see [F], sect. 4). Then the invariant  $T_K$  is not of finite type.

We omit the general proof but show this in an example for a class of degree 1. It is clear that one can find similar examples for any class of degree at least 1. *Example 3.4* We fix the system of generators  $\{\alpha, \beta\}$  of  $H_1(T^2; \mathbb{Z})$  as shown in Fig. 22.

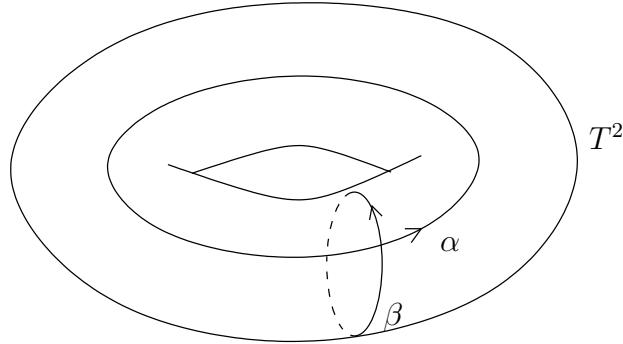


Fig. 22

Let  $f : T^2 \rightarrow S^1$  be a submersion such that the fibers represent  $\beta$ . The vector field  $v$  is defined as the unit tangent vector field to the fibers of  $f$ . Let us consider the family of knots  $K_n, n \in \mathbb{N}$  shown in Fig. 23.

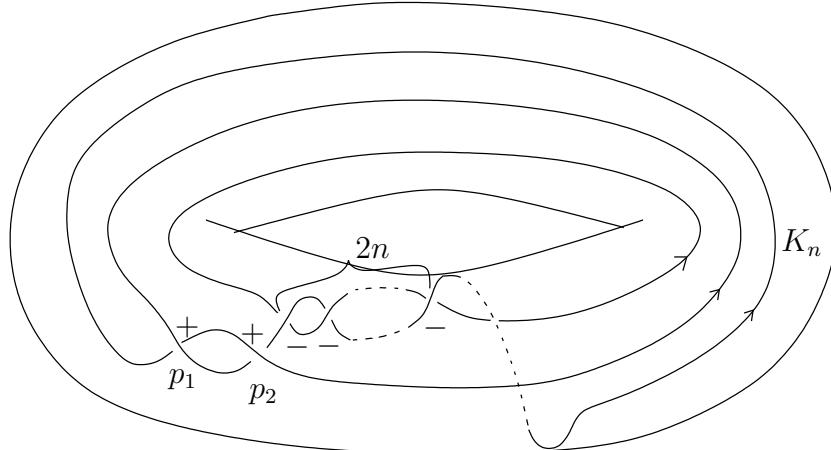


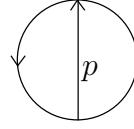
Fig. 23

$K_n \hookrightarrow T^2 \times \mathbb{R}$  is a global knot with respect to  $v$  and  $[K_n] = 3\alpha + \beta \in H_1(T^2)$ . We take as group  $G$ :

$$(H_1(T^2)/\langle [K_n] \rangle) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$$

Each knot  $K_n$  is  $G$ -pure, i.e. for each crossing  $p$  the loop  $pr(K_p^+)$  (as well as the loop  $pr(K_p^-)$ ) represents the non-trivial element in  $G \cong \mathbb{Z}/2\mathbb{Z}$ . (Thus, each chord is marked by the same element in  $G$  and we do not write the marking.)

Let us consider the unique  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration  $D$  of degree 1:



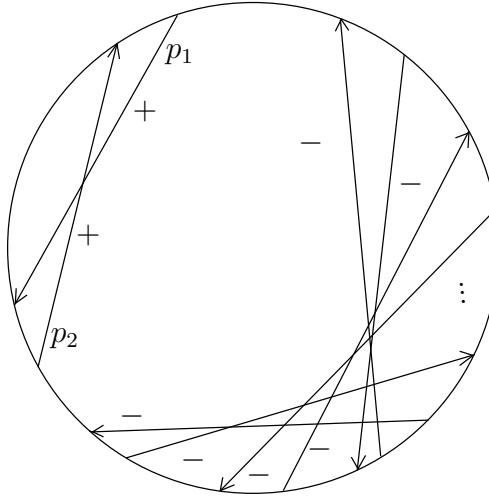
Let us consider the class of  $D$  of degree 1 defined by

$$c(D) = \sum_{\substack{\text{chords } p, q \\ \text{in } D}} w(q)$$

(One easily verifies that  $c(D)$  is indeed a class of  $D$ , because the chords  $p$  and  $q$  cannot get crossed in a  $\mathbb{Z}/2\mathbb{Z}$ -pure isotopy, and if a new couple of crossings  $q_i, i = 1, 2$

appears by a Reidemeister move of type  $II$ , then  $c(D_t)$  remains unchanged.)

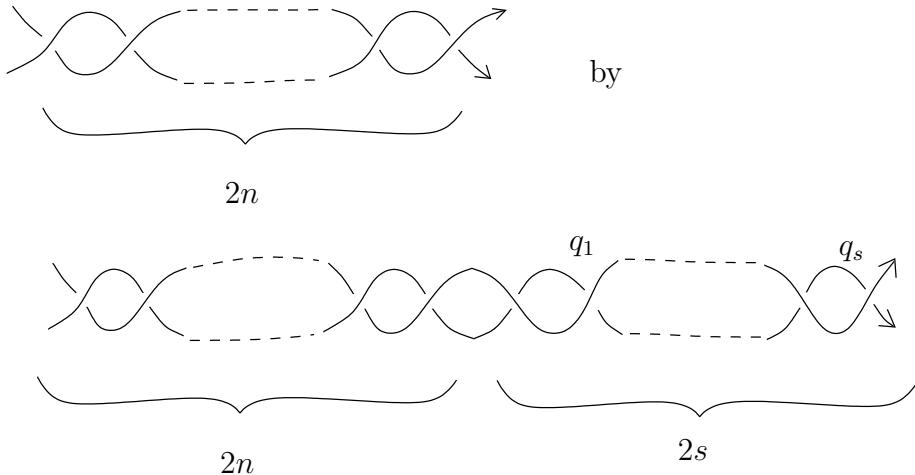
$K_n$  has the Gauss diagram depicted in Fig. 24.



**Fig. 24**

Consequently,  $c(p_1) = -n$ ,  $c(p_2) = 0$  and  $c(p_i) = 0$  or 1 for each crossing  $p_i$  of  $K_n \setminus \{p_1, p_2\}$ . Therefore,  $T_{K_n}(D; c(D) = -n) = +1$  and  $T_{K_n}(D; c(D) = r) = 0$  for all  $r \notin \{-n, 0, 1\}$ .

Let  $K'_n$  be any knot obtained from  $K_n$  by changing any of the  $2n$  crossings of  $K_n \setminus \{p_1, p_2\}$ . Then  $c(p_1)$  becomes strictly bigger than  $-n$  and thus,  $T_{K'_n}(D; c(D) = -n) = 0$ . This shows that no linear combination with coefficients  $\pm 1$  of  $T_{K_n}(D; c(D) = -n)$  with  $T_{K'_n}(D; c(D) = -n)$  can ever be equal to 0. Moreover, replacing the fragment in  $K_n$  depicted in the left-hand side of Fig. 25 by the one depicted in the right-hand side of Fig. 25 does not change the knot  $K_n$ . Switching any of the  $s$  crossings  $\{q_1, \dots, q_s\}$  makes  $c(p_1)$  strictly bigger than  $-n$ . Therefore,  $T_{K_n}(D; c(D) = -n)$  is of degree at least  $2n + s + 1$  and hence it is not of finite type. ■



**Fig. 25**

Clearly,  $T$ -invariants which are not of finite type cannot be extracted from the generalized Kontsevitch integral (which is the universal invariant

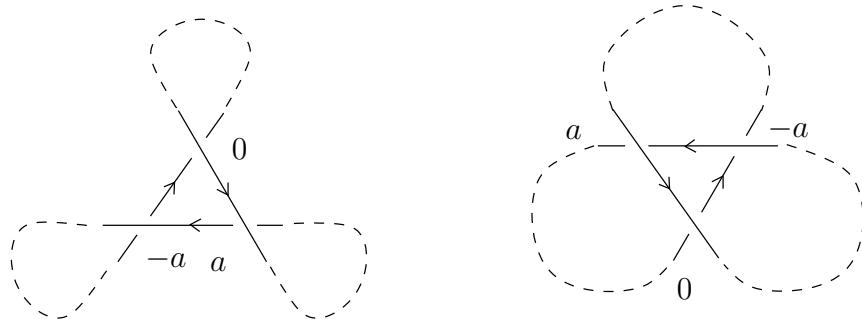
of finite type). Moreover, if we restrict ourselves to  $G$ -pure global knots, then the Kontsevitch integral is no longer even the universal invariant of finite type for these knots.

*Exemple 2.5* Let  $a \in G$  and  $a \notin \{0, \pm [K]_G\}$ . Let

$$D = \begin{array}{c} a \\ \diagup \quad \diagdown \\ p_1 \quad p_2 \end{array} , \quad n=0$$

$[K]_G - a$

$T_K(D; \emptyset)$  is a Gauss diagram invariant of degree 2 for  $G$ -pure global knots  $K \hookrightarrow T^2 \times \mathbb{R}$ . One easily verifies that  $T_K(D; \emptyset)$  is of degree 2 as a finite type invariant too. If in a (not  $G$ -pure) isotopy  $K_t$ , the knot  $K$  crosses exactly once a stratum of type e.g.  $a \circlearrowleft (0|a, -a)$  (see Fig. 26)



**Fig. 26**

then  $T_{K_t}(D; \emptyset)$  changes by  $\pm 1$ .

Consequently,  $T_K(D; \emptyset)$  is not invariant for *all* isotopies of  $K$  and therefore cannot be extracted from the generalized Kontsevitch integral.

**Remarks:**

1. The above invariant  $T_K(D; \emptyset)$  could have been equally considered as an invariant  $T_K(\emptyset; c(D))$  with

$$c(D) = \sum_{\substack{a \\ \diagup \quad \diagdown \\ p_1 \quad p_2}} w(q_1)w(q_2)$$

$[K]_G - a$

2. Let  $n = 0$  and let

$$D = \begin{array}{c} [K]_G - a \\ \text{or} \\ \text{Diagram} \end{array} \quad \quad D = \begin{array}{c} a \\ \text{Diagram} \\ a \end{array} \quad .$$

The corresponding  $T$ -invariants  $T_K(D; \emptyset)$  are the only other invariants of degree 2 which are of finite type and which cannot be extracted from the generalized Kontsevitch integral for knots in  $F^2 \times \mathbb{R}$ .

More generally, let us consider  $T$ -invariants of finite type under  $G$ -pure isotopy from the point of view of the works [V], [K], [BN]. In the case of knots in  $\mathbb{R}^3$ , the famous theorem of Kontsevitch states that each  $\mathbb{C}$ -valued functional on the  $\mathbb{C}$ -vector space of (unmarked) chord diagrams can be integrated to a knot invariant of finite type if it verifies the 1- $T$  and 4- $T$  relations. (It is an easy matter to see that these relations are necessarily verified by each invariant of finite type).

Let  $\mathcal{K}$  be a fixed free homotopy class of an oriented loop in  $F^2$  and let  $[K] \in H_1(F^2; \mathbb{Z})$  be the corresponding homology class. Let  $M_{\mathcal{K}}$  be the space of all possibly singular knot diagrams  $K \hookrightarrow F^2 \times \mathbb{R}$  such that  $pr(K)$  represents  $\mathcal{K}$ . Let  $G$  be a fixed quotient group of  $H_1(F^2; \mathbb{Z})/\langle [K] \rangle$ . Finally, let  $M_{\mathcal{K}}^G \hookrightarrow M_{\mathcal{K}}$  be the subspace of all possibly singular  $G$ -pure diagrams. Here, the marking in  $G$  of a double point of  $K$  is given by the corresponding marking of the positive resolution (see Fig. 27)

$$\begin{array}{ccc} \begin{array}{c} \text{Diagram} \\ \text{with} \\ \text{double} \\ \text{point} \end{array} & \longrightarrow & \begin{array}{c} \text{Diagram} \\ \text{with} \\ \text{double} \\ \text{point} \end{array} \end{array}$$

Fig. 27

(Near each crossing, we write the corresponding marking in  $G$ .) Evidently,  $M_{\mathcal{K}}$  is connected. We do not know whether  $M_{\mathcal{K}}^G$  is always connected or not. However, if we take out the set  $\Sigma$  of all singular diagrams in  $M_{\mathcal{K}}^G$ , then there are in general different components of  $M_{\mathcal{K}}^G \setminus \Sigma$  which represent the same knot type in  $F^2 \times \mathbb{R}$  (see sect. 9). Theorem 1 claims that for global knots,  $T$ -invariants do not depend on the chosen component of  $M_{\mathcal{K}}^G \setminus \Sigma$  for a given knot type in  $F^2 \times \mathbb{R}$ . But let us forget global knots for one moment, and let us consider  $G$ -pure knots only up to  $G$ -pure isotopy.

Let us have a look at the analogue of the above mentioned relations. Let  $\mathcal{A}_G$  be the  $\mathbb{C}$ -vector space generated by all chord diagrams with homological markings in  $G$  of the chords, and with the homotopical marking  $\mathcal{K}$  for the whole circle. Finally, let  $\mathcal{A}_G^0 \hookrightarrow \mathcal{A}_G$  be the subspace generated by the  $G$ -pure chord diagrams (i.e. there are no chords with marking  $0 \in G$ ). Let  $I : \mathcal{A}_G \rightarrow \mathbb{C}$  be a functional. We want to integrate it to a knot invariant.

- *1 – T relation:* This relation is obtained by going in  $M_{\mathcal{K}}$  around a diagram which has a double point (as a singular knot) in a cusp of the projection into  $F^2$ . See Fig. 28.

$$I \left( \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \right) - I \left( \begin{array}{c} \diagdown \diagup \\ 0 \end{array} \right) = 0$$

**Fig. 28**

- *2 – T relation:* We have this additional relation because, instead of considering only embeddings  $S^1 \hookrightarrow F^2 \times \mathbb{R}$ , we consider diagrams, i.e. embeddings together with the projection onto  $F^2$ . Then, the relation is obtained by going in  $M_{\mathcal{K}}$  around a diagram which has two double points in an autotangency of the projection (see Fig. 29).

$$I \left( \begin{array}{c} \diagup \diagdown \\ a \end{array} \right) - I \left( \begin{array}{c} \diagdown \diagup \\ a \end{array} \right) = 0$$

**Fig. 29**

(The markings in all other crossings or double points of these fragments are determined by the single marking  $a \in G$ .)

There are two cases to consider, which are shown in Fig. 30 and 31.

$$\left( I \left( \begin{array}{cc} \nearrow \curvearrowleft & \searrow \\ a & a \end{array} \right) - I \left( \begin{array}{cc} \nearrow \curvearrowright & \searrow \\ -a & a \end{array} \right) \right) -$$

$$\left( I \left( \begin{array}{cc} \nearrow \curvearrowleft & \searrow \\ -a & -a \end{array} \right) - I \left( \begin{array}{cc} \nearrow \curvearrowright & \searrow \\ -a & a \end{array} \right) \right) = 0$$

**Fig. 30**

$$\left( I \left( \begin{array}{cc} \nearrow \curvearrowright & \searrow \\ a & -a \end{array} \right) - I \left( \begin{array}{cc} \nearrow \curvearrowleft & \searrow \\ -a & -a \end{array} \right) \right) -$$

$$\left( I \left( \begin{array}{cc} \nearrow \curvearrowleft & \searrow \\ a & -a \end{array} \right) - I \left( \begin{array}{cc} \nearrow \curvearrowright & \searrow \\ a & a \end{array} \right) \right) = 0$$

**Fig. 31**

- *4 – T relation:* This relation is obtained by going in  $M_K$  around a diagram with three double points in a triple point of the projection. For each  $a, b \in G$ , we

have the relation shown in Fig. 32.

$$\begin{aligned}
 & I \left( \text{Diagram 1} \right) - I \left( \text{Diagram 2} \right) \\
 & + I \left( \text{Diagram 3} \right) - I \left( \text{Diagram 4} \right) = 0
 \end{aligned}$$

Diagrams 1, 2, 3, and 4 are chord diagrams on a torus. Each diagram consists of a dashed circle with two points on the boundary labeled  $a$  and  $b$ . The diagrams show different configurations of chords connecting these points, with arrows indicating orientation. Diagram 1 has a chord  $a-b$  and a crossing. Diagram 2 has a chord  $b-a$  and a crossing. Diagram 3 has a chord  $b-a$  and a crossing. Diagram 4 has a chord  $b-a$  and a crossing.

**Fig. 32**

The proofs are completely analogous to the one for knots in  $\mathbb{R}^3$  (see also [Go] where it is done for knots in the solid torus). Each functional  $I$  which can be integrated to a knot invariant verifies  $1 - T$ ,  $2 - T$ ,  $4 - T$ .

*Question: Can each functional which verifies  $1 - T$ ,  $2 - T$ ,  $4 - T$  be integrated to a knot invariant?*

**Remark.** Goryunov [Go] has shown that the answer is "yes" in the case of the solid torus. Notice that our chord diagrams are planar and with homological markings. In [A-M-R] it is shown that the answer to the above question is "yes" if  $\partial F^2 \neq \emptyset$  and if one uses chord diagrams which are immersed in  $F^2$  instead of planar chord diagrams with homological markings. But then, it seems to be difficult to find such functionals.

Let us consider now functionals  $I^0 : \mathcal{A}_G^0 \rightarrow \mathbb{C}$ . Evidently, if  $I^0$  can be integrated to a knot invariant under  $G$ -pure isotopy, then it has to verify the corresponding relations  $1 - T^0 = \emptyset$ ,  $2 - T^0$ ,  $4 - T^0$  similar to the previous ones but where the marking  $0 \in G$  is forbidden. *par Example 3.6*

$$I_1^0 := T_K \left( \text{Diagram} ; \emptyset \right), \quad a \neq 0$$

Diagram is a chord diagram on a torus with two points on the boundary labeled  $a$  and  $-a$ . It consists of a circle with two vertical chords. The top chord is labeled  $a$  and the bottom chord is labeled  $-a$ .

$I_1^0$  verifies  $1 - T$ ,  $2 - T$ , and  $4 - T^0$  but it does *not* always verify  $4 - T$  (see Fig. 33).

$$\begin{aligned} & I \left( \begin{array}{c} \text{Diagram 1: A circle with a central crossing labeled '0'. Two strands labeled 'a' enter from the left, and two strands labeled 'a' exit to the right.} \end{array} \right) - I \left( \begin{array}{c} \text{Diagram 2: A circle with a central crossing labeled '0'. Two strands labeled 'a' enter from the left, and two strands labeled 'a' exit to the right.} \end{array} \right) \\ & + I \left( \begin{array}{c} \text{Diagram 3: A circle with a central crossing labeled '0'. Two strands labeled 'a' enter from the left, and two strands labeled 'a' exit to the right.} \end{array} \right) - I \left( \begin{array}{c} \text{Diagram 4: A circle with a central crossing labeled '0'. Two strands labeled 'a' enter from the left, and two strands labeled 'a' exit to the right.} \end{array} \right) = -1 \end{aligned}$$

Fig. 33

Indeed, the only non-zero contribution to  $I_1^0$  from two of the three involved crossings comes from the term shown in Fig. 34, and is equal to  $w(p)w(q) = -1$ .

$$- \left( -I_1^0 \left( \begin{array}{c} \text{Diagram 2: A circle with a central crossing labeled '0'. Two strands labeled 'a' enter from the left, and two strands labeled 'a' exit to the right.} \end{array} \right) \right)$$

Fig. 34

*Exemple 3.7*

$$I_2^0 := T_K \left( \begin{array}{c} a \\ \text{Diagram: A circle with two strands labeled 'a' entering from the left and two strands labeled 'a' exiting to the right.} \\ a \end{array} ; \quad \emptyset \right)$$

with  $a \neq 0$  and  $a \neq -a$  in  $G$ .  $I_2^0$  verifies  $1 - T$  and  $4 - T^0$  but it does *not* always verify  $2 - T^0$ ! Indeed, in case 2, we have the relation shown in

Fig. 35.

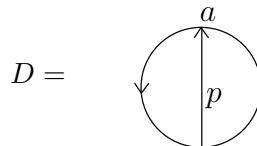
$$\left( I_2^0 \left( \begin{array}{cc} \nearrow \swarrow & \\ a & -a \end{array} \right) - I_2^0 \left( \begin{array}{cc} \nearrow \swarrow & \\ -a & -a \end{array} \right) \right) -$$

$$\left( I_2^0 \left( \begin{array}{cc} \nearrow \swarrow & \\ a & -a \end{array} \right) - I_2^0 \left( \begin{array}{cc} \nearrow \swarrow & \\ a & a \end{array} \right) \right) = -1$$

**Fig. 35**

However,  $I_2^0$  is an isotopy invariant for  $G$ -pure global knots because autotangencies with opposite directions of the branches do not occur in an isotopy through global knots. par Question. If a functional  $I^0 : \mathcal{A}_G^0 \rightarrow \mathbb{C}$  verifies  $2 - T^0$  and  $4 - T^0$ , can it be integrated to a knot invariant under  $G$ -pure isotopy?

Let us consider now Gauss diagram invariants of degree 2 for global knots  $K$ , which are not of finite type. From now on,  $G$  will always be a quotient group of  $H_1(F^2; \mathbb{Z})/\langle [K] \rangle$ . Hence, a knot is  $G$ -pure if and only if there is no marking equal to 0 in  $G$ . By the above lemmas and the definition of the degree of a Gauss diagram invariant, we must have  $m = n = 1$ . Consequently,



for some  $a \in G$ ,  $a \neq 0$ .

**Definition 3.7**

$$c_{++}(D) = \sum \quad w(q) \quad \quad c_{--}(d) = \sum \quad w(q)$$

$$c_{+-}(D) = \sum w(q) \quad c_{-+}(D) = \sum w(q)$$

**Lemma 3.3**  $c_{++}(D), c_{--}(D), c_{+-}(D), c_{-+}(D)$  are  $G$ -pure classes of degree 1 of  $D$ . Moreover, these are the only  $G$ -pure classes of degree 1.

**Proof.** Evidently,  $c_{++}, c_{--}, c_{+-}, c_{-+}$  are  $G$ -pure classes of  $D$  because  $p$  and  $q$  cannot get crossed in a  $G$ -pure isotopy. Assume now that  $p$  and  $q$  are crossed. None of the configurations depicted in Fig. 36

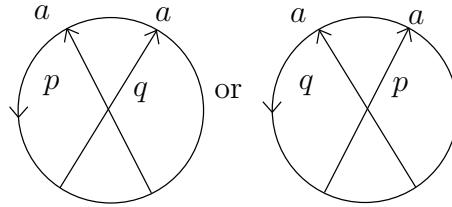


Fig. 36

enters into the class. Indeed, if one of them did, it would be invariant under Reidemeister moves of type  $II$ . Assume that the configuration in the left-hand side of Fig. 37

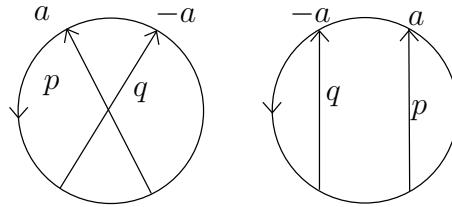


Fig. 37

enters into the class. The stratum  $a_0(a| -a, 2a)$  of the discriminant forces then the configuration in the right-hand side of Fig. 37 to enter into the class too. But then, the stratum  $a_0^+(a|2a, -a)$  forces the configuration in

the left-hand side of Fig. 38

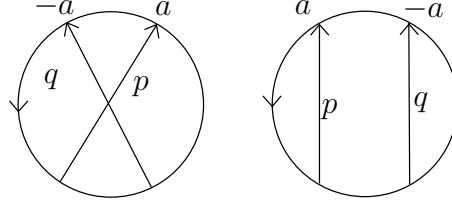


Fig. 38

to be also in the class. Now, the stratum  $a^+_G(a|2a, -a)$  forces the configuration in the right-hand side of Fig. 38 to be in the class too. Thus, *all* crossings with marking  $-a$  enter into the class. This class is therefore equal to  $W_K(-a)$  independently of  $p$  (see [F] for the definitions of the strata and of  $W_K(-a)$ ). Therefore, the corresponding  $T$ -invariant would be  $W_K(a) \cdot W_K(-a)$ , which is not new, and is of course of finite type. ■

**Proposition 3.1**

$$T_K := T_K \left( D = \begin{array}{c} a \\ \text{---} \\ \text{---} \end{array} \right) ; \quad c_{++}(D) = c_1, \quad c_{--}(D) = c_2, \quad c_{+-}(D) = c_3, \quad c_{-+}(D) = c_4$$

is the universal  $T$ -invariant of degree 2 which is not of finite type for  $G$ -pure global knots.

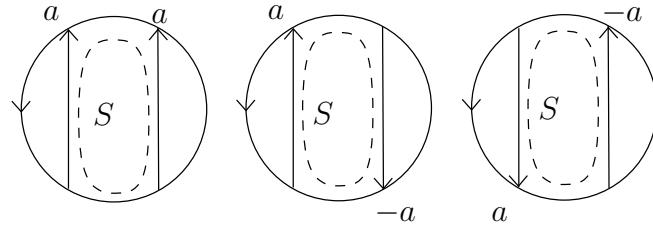
(Compare with Definition 3.4.)  $T_K$  is "universal" means that any other invariant (of degree 2, not of finite type) can be extracted from  $T_K$ .

**Proof.** The proposition is an immediate consequence of Theorem 1 and Lemma 3.3. ■ In sect. 9, we show an application of the above invariant. Let  $K_1, K_2 \hookrightarrow F^2 \times \mathbb{R}$  be two global knots with respect to the same non-elliptic vector field  $v$  on  $F^2$ . We assume that  $K_i, i = 1, 2$  are not solid torus knots in  $F^2 \times \mathbb{R}$ . Let  $\mathcal{G}$  be the set of all possible quotient groups  $G$  of  $H_1(F^2, \mathbb{Z})$  such that  $K_1$  and  $K_2$  have global representatives which are  $G$ -pure. Let  $\mathcal{T}$  be the set of all  $T$ -invariants of  $G$ -pure knots with respect to some  $G \in \mathcal{G}$ .

**Conjecture:** *If  $K_1$  and  $K_2$  are not isotopic, then there are  $T$ -invariants in  $\mathcal{T}$  which distinguish them.*

**Remarks:**

1. Remember that the usual invariants of finite type, in particular the free homotopy class of the knot, are a subset of  $T$ -invariants for  $m = 0$ .
2. For solid torus knots, the  $T$ -invariants are nothing but the usual invariants of finite type (extracted from the generalized Kontsevitch integral). Indeed,  $H_1(S^1 \times I; \mathbb{Z})/\langle [K] \rangle$  is a finite group  $G$ . In the Gauss diagram of  $K$  with markings  $a, -a$  in  $G$ , there are no subdiagrams of the forms depicted in Fig. 39 at all (because  $s$  would be a global knot homologous to 0).



**Fig. 39**

Hence, there are no specific  $T$ -invariants with respect to  $G$ . Therefore, we have to consider a quotient  $G'$  of  $G$ . This means that some  $a \neq 0 \in G$  becomes 0 in  $G'$ . But, if for a closed braid  $K$  and a given  $a \in G$ , no crossing is of type  $[K_p^+] = a$ , then it is easily seen that there always exist a crossing  $p$  with  $[K_p^+] = -a \in G$ . But this means that  $K$  is never a  $G'$ -pure global knot and, hence, there are not any specific  $T$ -invariants.

3. Global solid torus knots are closed braids. They are classified by Artin's theorem together with Garside's solution of the conjugacy problem in braid groups. Unfortunately, this solution has exponential complexity.

## 4 $T$ -invariants separate $\mathbb{Z}/2\mathbb{Z}$ -pure global knots in $T^2 \times \mathbb{R}$

Let  $\{\alpha, \beta\}$  be generators of  $H_1(T^2)$  as shown in Fig. 22. It is more convenient to use the non-generic vector field  $v$  which is tangent to the fibers of  $f$  (see Example 3.4). The difference with a generic vector field (obtained by a small perturbation of  $v$ ) is that for the latter, positive multiples of  $\beta$  can be represented by global knots. But in any case, these are solid torus knots and we do not consider them.

**Definition 4.1** A global knot  $K \hookrightarrow T^2 \times \mathbb{R}$  with respect to  $v$  is called a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot if:

1.  $H_1(T^2; \mathbb{Z})/\langle [K] \rangle \cong \mathbb{Z}$
2.  $K$  is  $\mathbb{Z}/2\mathbb{Z}$ -pure for  $G = \mathbb{Z}/2\mathbb{Z} \cong (H_1(T^2)/\langle [K] \rangle)/2\mathbb{Z}$

**Remark.** In particular, condition 1 implies that a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot  $K$  is a solid torus knot if and only if  $K \hookrightarrow T^2$  (i.e.  $K$  is a torus knot). We show a typical example of a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot in Fig. 40.  $[K] = 3\alpha + \beta$  and  $\alpha$  is a generator of  $H_1(T^2)/\langle [K] \rangle$ . Notice that switching a crossing  $p$  does not change the marking of  $p$  for a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot. Consequently, the property of a global knot to be  $\mathbb{Z}/2\mathbb{Z}$ -pure or not depends only on the underlying curve  $pr(K) \hookrightarrow T^2$ .

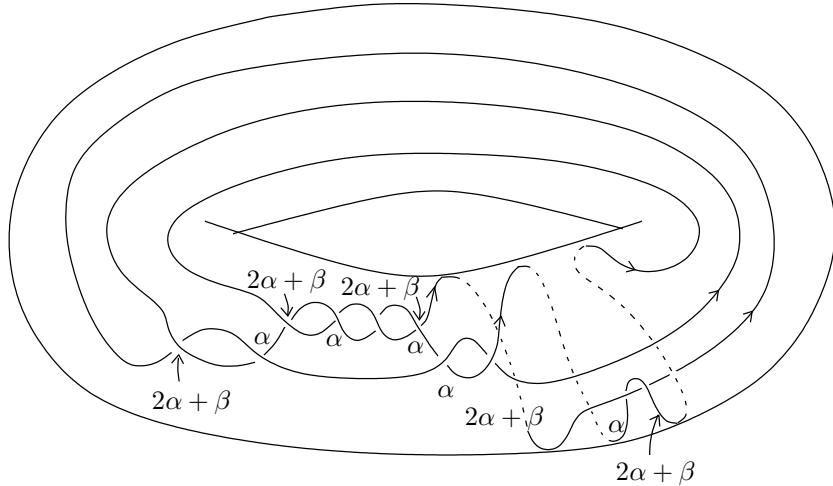


Fig. 40

**Theorem 4.1** *Let  $K \hookrightarrow T^2 \times \mathbb{R}$  be a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot with  $c$  crossings. Let  $d$  be any natural number not bigger than  $c$ . Then, the knot  $K$  is uniquely determined by the set of all  $T$ -invariants  $T_K$  of finite type, of the degrees  $(m = d, n = 0)$  with respect to  $G = \mathbb{Z}/2\mathbb{Z}$ .*

**Remark.** The number of such  $T$ -invariants is finite. Thus, Theorem 2 proves the conjecture in sect. 3, in the case of  $\mathbb{Z}/2\mathbb{Z}$ -pure global knots in  $T^2 \times \mathbb{R}$  and, moreover gives an effective solution to the problem. Notice that we do not need here the  $T$ -invariants of infinite type.

**Proof.** The proof consists of two steps.

*Step 1:* The Gauss diagram with markings in  $G \cong \mathbb{Z}/2\mathbb{Z}$  determines  $K$ .

*Step 2:* The invariants  $T_K$  determine the Gauss diagram of  $K$ .

*Step 1:* For local knots, it is well known and rather evident. But it is not at all obvious for knots in  $T^2 \times \mathbb{R}$ . By definition, the marking of each crossing of a  $\mathbb{Z}/2\mathbb{Z}$ -pure knot is the non-trivial element in  $\mathbb{Z}/2\mathbb{Z}$ . Therefore, we do not write it in Gauss diagrams, configurations, etc ... The  $T$ -invariant of degree  $(d_1, d_2) = (0, 0)$  is the free homotopy class of  $K$ , or (equivalently here), the homology class represented by  $K$  (which is a primitive class because  $K$  is not a solid torus knot). Consequently, we have to show that the Gauss diagram determines  $K$  in its given homology class.

**Definition 4.2** A set of arrows in a Gauss diagram is called a *bunch of arrows* if their number is even and:

1. they are near to each other (i.e. there are small arcs on  $S^1$  between them where no other arrow starts or ends)
2. each two arrows cut in exactly one point
3. the orientation of the arrows is alternating
4. all the arrows have the same writhe

(See an example in Fig. 41.)

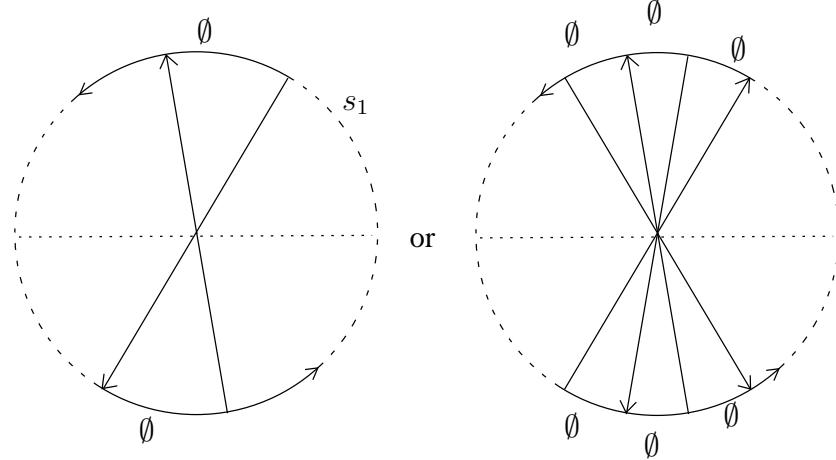


Fig. 41

**Lemma 4.1** *After possibly performing Reidemeister moves of type II, such that each of them decreases the number of crossings (see left-hand part of Fig. 42),*

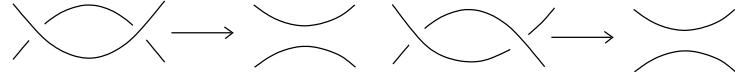
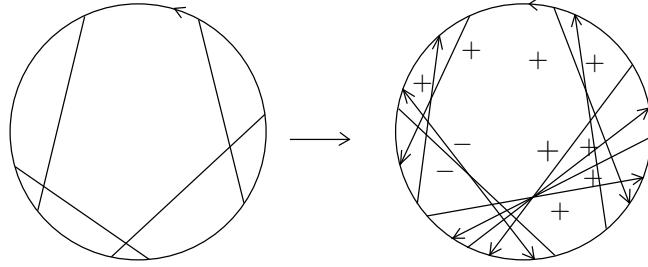


Fig. 42

*the Gauss diagram of a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot is of the following form: some chord diagram where each chord is replaced by some bunch of arrows. Moreover, there exists a homotopy from  $K$  to a torus knot  $K'$  which is an isotopy of diagrams besides possibly performing transformations of the type depicted in the right-hand part of Fig. 42.*

(Of course,  $K'$  is determined by the homology class represented by  $K$ .)

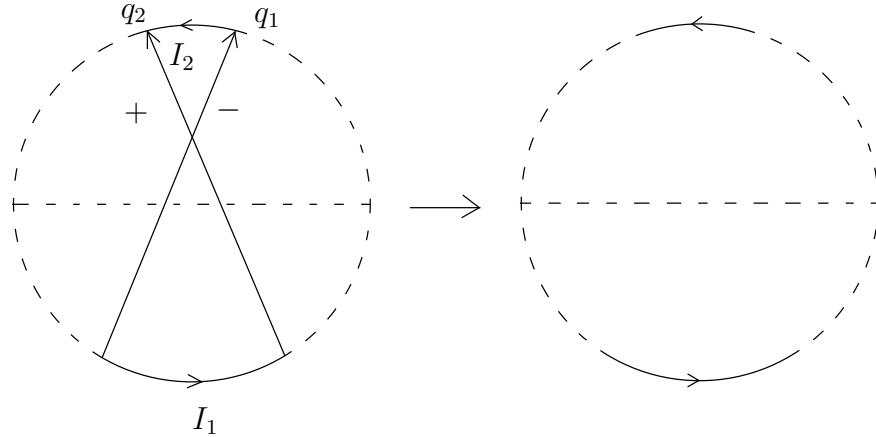
*Example* The knot in Fig. 40 corresponds to the diagram in the right-hand part of Fig. 43.



**Fig. 43**

Notice that e.g.  can never appear as the underlying chord diagram of a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot. (This is an exercise).

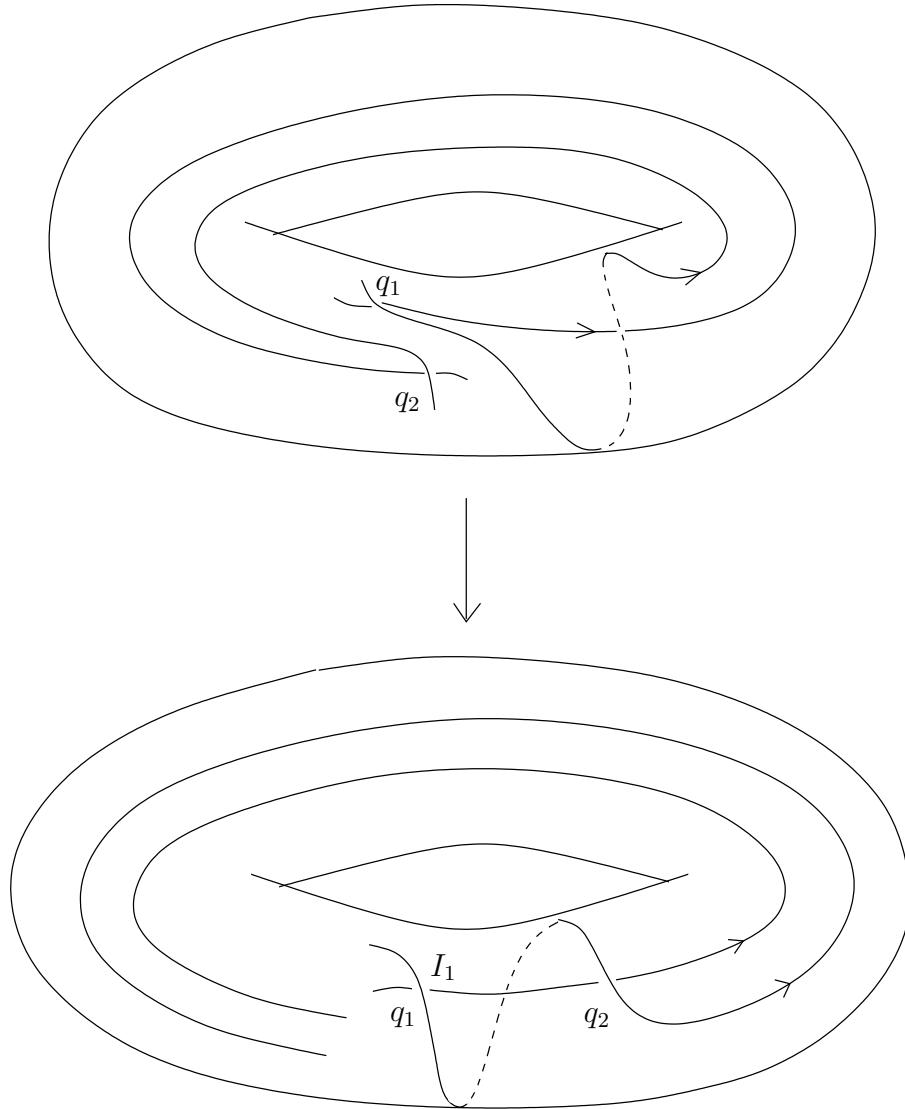
**Proof of Lemma 4.1** We start by eliminating all couples of arrows corresponding to crossings say  $q_1$  and  $q_2$  as shown in Fig. 44.



**Fig. 44**

We assume that the arcs  $I_1$  and  $I_2$  are empty. We have to prove that both arcs  $I_1$  and  $I_2$  are small or equivalently that  $[K_{q_1}^+] = [K_{q_2}^+] \in H_1(T^2; \mathbb{Z})$ . Then the above operation corresponds to a Reidemeister move inverse to the one depicted in the left-hand part of Fig. 42. After possibly performing a (global) isotopy of the diagram, we may assume that at least one of the

two arcs, say  $I_1$ , is small (see Fig. 45).



**Fig. 45**

If the arc  $I_2$  cannot be made small at the same time as  $I_1$ , then we have exactly the situation shown in Fig. 45. But then  $w(q_1) = w(q_2)$  in contradiction to our assumption on  $q_1$  and  $q_2$ . We observe now that for a diagram of a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot, there are no possible Reidemeister moves of type

*III* at all. Indeed, in  $pr(K)$ , there cannot be any triangle

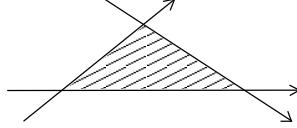


Fig. 46

as shown in Fig. 46 because the three crossings  $q_i, i = 1, 2, 3$  always verify a relation:  $[K_{q_3}^+] = [K_{q_1}^+] + [K_{q_2}^+] \text{mod}[K]$ . Consequently, they could not be all three non-zero in  $G \cong \mathbb{Z}/2\mathbb{Z}$ . After having performed all possible Reidemeister moves of type *II* as in the left-hand side of Fig. 42, we obtain a diagram of  $K$  which we call *minimal*. It is characterized by the fact that it does not allow any Reidemeister moves, except for those which strictly increase the number of crossings (i.e. the move inverse to the one in the left-hand side of Fig. 42).

**Claim 1.** If the minimal diagram is not empty, then it contains always a *2-gon* (see the left-hand part of Fig. 47).

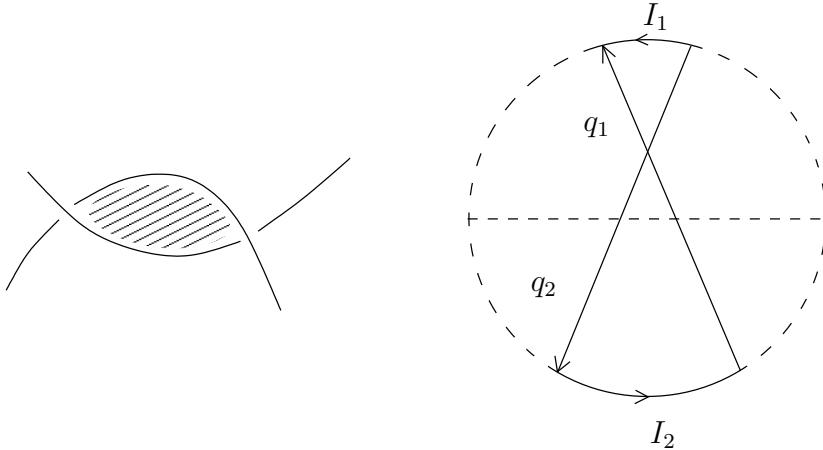


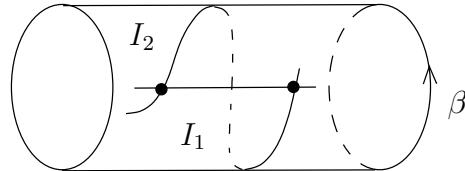
Fig. 47

Moreover, the following fragment (Fig. 47, right-hand side) of the minimal Gauss diagram of a  $\mathbb{Z}/2\mathbb{Z}$ -pure knot corresponds always to a 2-gon. Here,  $w(q_1) = w(q_2)$  and  $I_i, i = 1, 2$  are both empty.

**Proof of the claim.** We start with the following observation: Let  $D(K)$  be a knot which is obtained from  $K$  after performing a Dehn twist of  $T^2$ . We do not change the vector field  $v$ . If the Dehn twist is along  $\beta$ , or

positive along  $\alpha$ , then  $D(K)$  is still a global ( $\mathbb{Z}/2\mathbb{Z}$ -pure) knot with respect to  $v$  (see Fig. 22). By definition, the two sides of a 2-gon form a loop which is homotopic to 0 in  $T^2$ . Let us consider first "fake" 2-gons, i.e. 2-gons in  $pr(K) \subset T^2$  such that the corresponding loop is not homotopic to 0 in  $T^2$ . Using the above observation, we can restrict our considerations exactly to the two cases shown in Fig. 48.

Case 1



Case 2

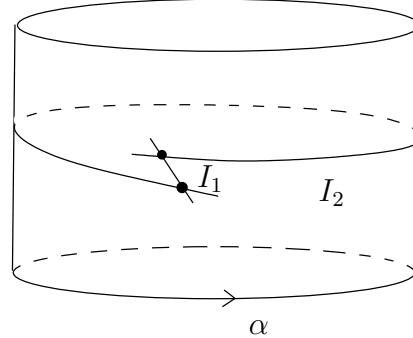


Fig.48

$I_1 \cup I_2$  does not cut the rest of the knot  $K$ . Consequently, in the second case,  $K$  is a solid torus knot which is not a torus knot (the minimal diagram is not empty). We do not consider these knots. In the first case,  $K$  represents  $2\alpha + x\beta, x \in \mathbb{Z}$  and  $x$  odd. One easily checks that in the minimal diagram of  $K$  there is always some 2-gon. An example is shown in Fig. 49.

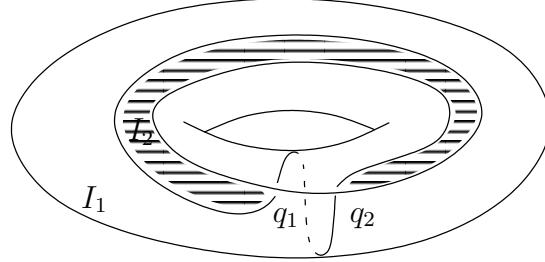
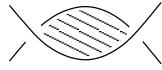
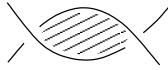


Fig. 49

If we change in the diagram in Fig. 47 exactly one of the crossings  $q_1$  or  $q_2$  to its inverse, then we obtain a fragment as shown in Fig. 44. We have already proven that the fragment in Fig. 44 corresponds to



Consequently, the fragment in Fig. 47 corresponds to



which is a 2-gon.

Assume now that the diagram of  $K$  (in general position) contains no 2-gons at all. We have already proven that the diagram of  $K$  contains no triangles whose three sides form a loop which is contractible in  $T^2$ .

**Sub-claim.** *The sides of each  $n$ -gon in  $pr(K) \subset T^2$  form a contractible loop if  $n \geq 3$ .*

Indeed, either we are in the situation analogue to case 2 in Fig. 48, and hence,  $K$  is a solid torus knot, or we are in the situation analogue to case 1 in Fig. 48. But this is not possible if  $n \geq 3$  as shown in Fig. 50:

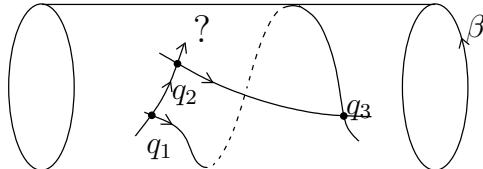


Fig. 50

The branch of  $K$  through  $q_1$  and

$q_2$  is blocked. This proves the subclaim.

The assumption (no 2-gons) together with the subclaim imply that the

4-valent graph  $pr(K) \subset T^2$  splits  $T^2$  into contractible

4-gons, 5-gons ... Let  $v_0$  be the number of vertices and  $v_1$

be the number of edges of  $pr(K)$ . Let  $v_2$  be the number of

components of  $T^2 \setminus pr(K)$ . Evidently,  $v_1 = 2v_0$ .

One has:  $v_0 - v_1 + v_2 = \chi(T^2) = 0$  and hence,  $v_0 = v_2$ .

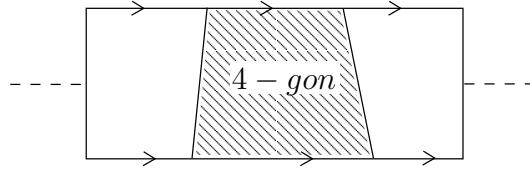
We denote by  $\#(\cdot)$  the number of  $(\cdot)$ .

$\#(\text{angles}) = 4v_0 = 4v_2$ . On the other hand,  $\#(\text{angles}) = 4\#(\text{4-gons}) + 5\#(\text{5-gons}) + \dots$

This implies that  $0 = \#(\text{5-gons}) = \#(\text{6-gons}) = \dots$

Consequently,  $T^2 \setminus pr(K)$  consists only of contractible

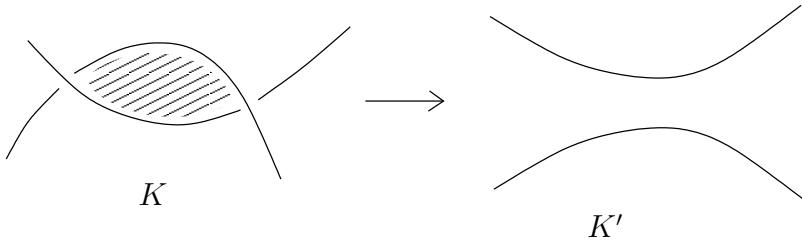
4-gons. We take one of them. It has at least two opposite sides which have the same orientation (induced by the orientation of  $K$ ). We add to the 4-gon the two neighbouring 4-gons corresponding to the remaining two sides (see Fig.51).



**Fig. 51**

We continue the process and at the end, we obtain an orientable immersed band in  $T^2$ . But the boundary of the band has two components contradicting the fact that  $K$  is a knot. This proves Claim 1.

We take now the existing 2-gon and make a homotopy as indicated in Fig. 52.

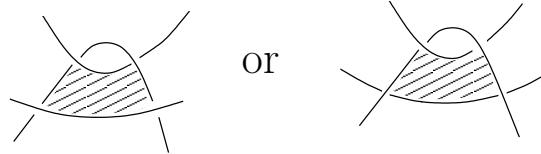


**Fig. 52**

$K'$  is again a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot.

**Claim 2** *If  $K$  was already minimal, then  $K'$  is minimal too.*

**Proof of Claim 2** If  $K'$  is not minimal, then it contains a fragment as shown in Fig. 53



**Fig. 53**

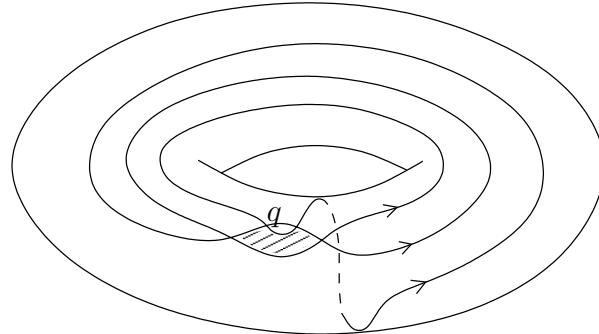
We have already proven that we can transform  $K$  into a torus knot by performing only the operations



on the diagram of  $K$ . Therefore, we

may assume that we have eliminated all crossings of  $K$  outside of the above fragment. Again, by using appropriate Dehn twists, we can reduce our considerations to the two cases shown in Fig. 54. (We need only  $pr(K) \subset T^2$ .)

Case 1



Case 2

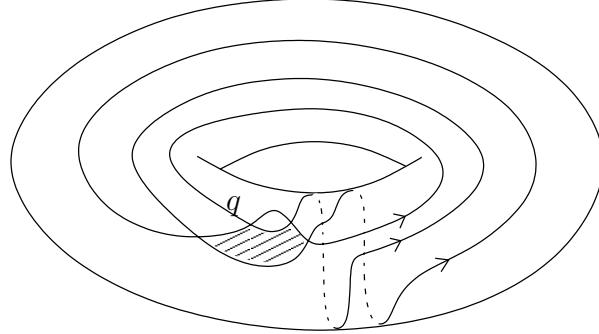
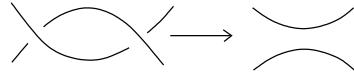


Fig. 54

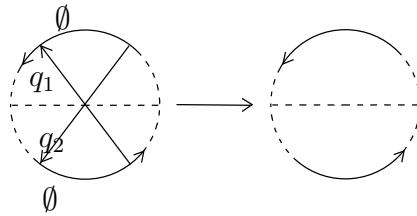
In case 1, we have  $[K_q^+] = 2\alpha \bmod [K]$  and in case 2, we have  $[K_q^+] = 2\alpha + 2\beta \bmod [K]$ . Consequently,  $[K_q^+] = 0 \in G$  and  $K$  was not  $\mathbb{Z}/2\mathbb{Z}$ -pure. This proves Claim 2.

By Claim 2, we can reduce the minimal diagram of  $K$  to a torus knot by using *only* the operation



Moreover, we have proven that

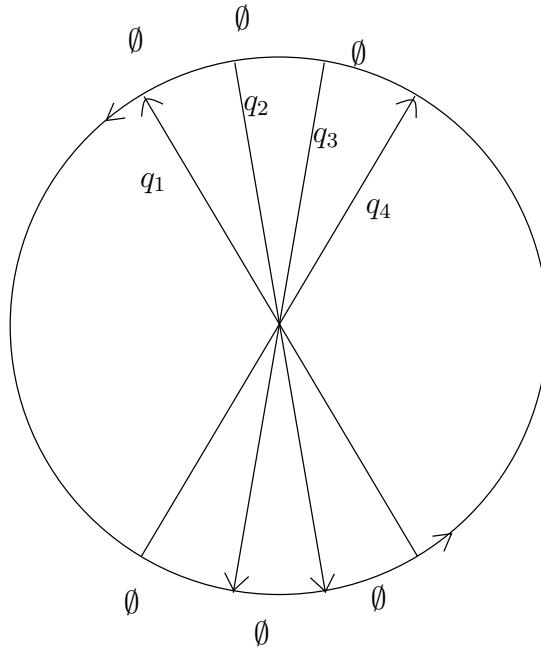
this operation corresponds exactly to the operation in Fig. 55.



$$w(q_1) = w(q_2)$$

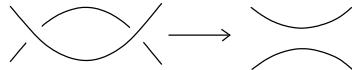
Fig. 55

Suppose, that we have a fragment as shown in Fig. 56.



**Fig. 56**

In fact, we have already shown that this implies automatically  $w(q_1) = w(q_2)$ ,  $w(q_3) = w(q_4)$ ,  $w(q_2) = -w(q_3)$ , and that  $q_2, q_3$  can be eliminated by a move



Consequently, repeating the operation



creates just bunches of arrows and Lemma 4.1 is proven.

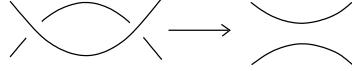
**Lemma 4.2** Let  $K$  be a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot.

Then,  $K$  is determinated by its Gauss diagram with markings in  $G \cong \mathbb{Z}/2\mathbb{Z}$  together with the homology class  $[K] \in H_1(T^2; \mathbb{Z})$ .

**Proof.**

As we have seen in the proof of Lemma 4.1, we can detect all possible

moves



with the Gauss diagram. Performing these moves, we obtain the minimal diagram of  $K$ . By Lemma 4.1, the minimal diagram of  $K$  is obtained from the torus knot  $K'$  (which is determinated by its homology class), by performing only operations



on the diagram of  $K'$ . Each such operation corresponds to a bunch of two arrows. Thus, we only need to show that the resulting knot is completely determined by the place of the bunch in the Gauss diagram, the directions of the arrows and their writhe. Indeed, the operation shown in Fig. 57

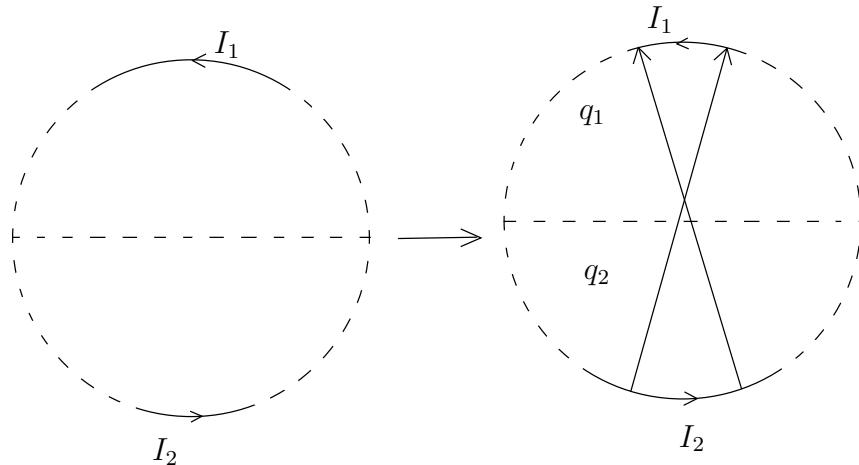


Fig. 57

corresponds to the change in Fig. 58

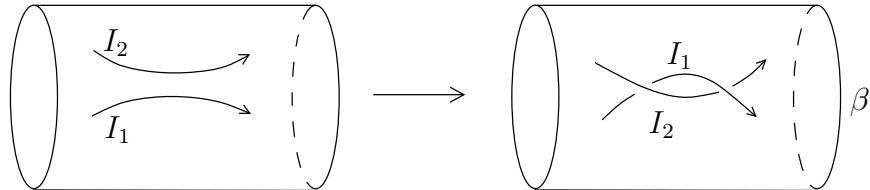


Fig. 58

if  $w(q_1) = w(q_2) = +1$ , or to the  
change in Fig. 59 if  $w(q_1) = w(q_2) = -1$ .

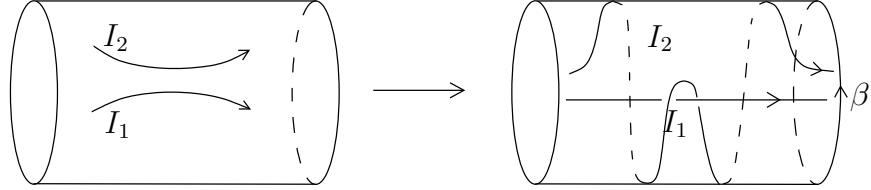


Fig. 59

Lemma 4.2 is proven.

Step 2

**Lemma 4.3** *Let  $K$  be a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot.*

A) *Let*

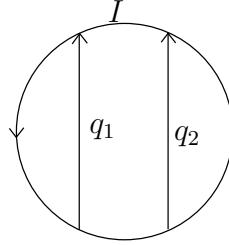


Fig. 60

*the diagram in Fig. 60 occur as subdiagram of the minimal diagram of  $K$  in such a way that  $I = \emptyset$  (i.e. there do not start or end any arrows in  $I$ ). Then  $w(q_1) = w(q_2)$ .*

*Let*

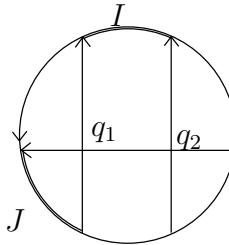


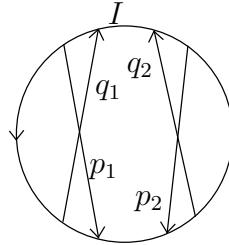
Fig. 61

*the diagram in Fig. 61 occur as subdiagram of the minimal Gauss diagram of  $K$  in such a way that  $I = \emptyset$ ,  $J = \emptyset$ . Then  $w(q_3) = -w(q_1) = -w(q_2)$ .*

**Remark.**

Lemma 4.1 implies that all three crossings belong to different bunches. Evidently, Lemma 4.3 allows to calculate all the writhes of a minimal diagram if one knows the writhe of *one* bunch of arrows.

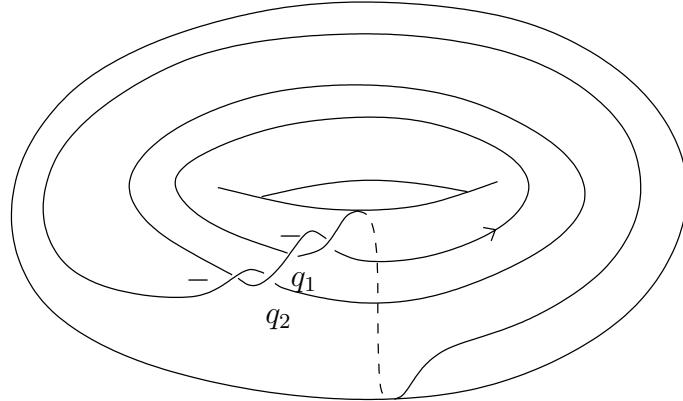
**Proof of Lemma 4.3.** We will prove only A). The proof of B) is similar, and is therefore omitted. Using Lemma 4.1, we can eliminate all crossings of  $K$ , except of the four crossings shown in Fig. 62.



**Fig. 62**

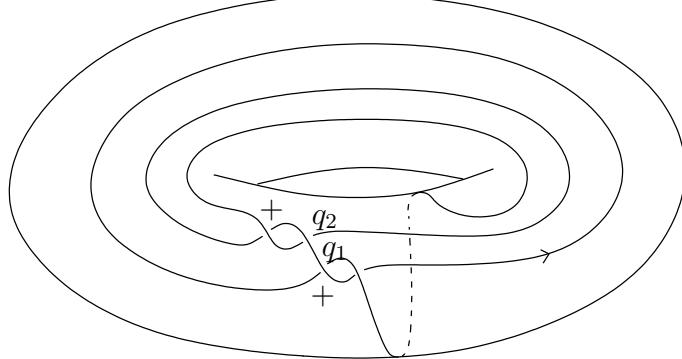
We know already that  $w(q_1) = w(p_1)$ ,  $w(q_2) = w(p_2)$ . After suitable Dehn twists, and after making  $I$  small,  $K$  is transformed into a knot  $K'$  so that one of the possibilities depicted in Fig. 63 is realized. In both cases,  $w(q_1) = w(q_2)$ . ■

Case 1



$$[K'] = 3\alpha + \beta$$

Case 2



$$[K'] = 3\alpha - \beta$$

**Fig. 63**

Let  $K$  be the diagram of a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot with  $c$  crossings and let  $K'$  be the corresponding minimal diagram with  $c'$  crossings. (We remind that a "diagram" is a knot together with his regular projection into  $T^2$ .) Each  $T$ -invariant  $T_K$  of degree  $(d, 0)$  is 0 for  $d > c'$ . Indeed,  $K$  is isotopic to  $K'$  and in the Gauss diagram of  $K'$ , there are not any configurations of  $d$  arrows. Let  $D$  be the Gauss diagram of  $K'$ , and let  $\bar{D}$  be the Gauss diagram of  $K'$ , without the writhes. Thus,  $\bar{D}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration of degree  $c'$  (see Def. 3.2). By Theorem 1,

$$T_K(\bar{D}; \emptyset) := \sum_{\bar{D}} w(p_1) \cdots w(p_{c'})$$

is an isotopy invariant of  $K$ . In each bunch, there is an even number of arrows, and, consequently,  $T_K(\bar{D}; \emptyset) = +1$ . By Lemma 4.3, there are only two possibilities for the writhes of  $D$ . Therefore, the  $T$ -invariant (of finite type)  $T_K(\bar{D}; \emptyset)$  of degree  $(c', 0)$  almost determines  $K$ : there are at most two knots with the same invariant. Their Gauss diagrams are obtained one from the other by a simultaneous switch of the writhes. Let  $K_1$  and  $K_2$  be the corresponding knots (we know already that they are determined by their Gauss diagrams). We have to distinguish them by  $T$ -invariants of smaller degree.

Let  $p$  be an arrow of  $\bar{D}$  and let  $\bar{D}_p$  be the configuration  $\bar{D} \setminus p$  of degree  $c' - 1$ . Of course, different  $p$  could determine the same configuration  $\bar{D}_p$ . Evidently, for each configuration  $\bar{D}_p$ , we have

$$T_{K_1}(\bar{D}_p; \emptyset) := \sum_{\bar{D}_p \subset D(K_1)} \prod_{p_i \in \bar{D}_p} w(p_i) = -T_{K_2}(\bar{D}_p; \emptyset)$$

Consequently, if  $T_K(\bar{D}_p; \emptyset) \neq 0$  for some  $\bar{D}_p$ , then  $T_K(\bar{D}; \emptyset)$  together with  $T_K(\bar{D}_p; \emptyset)$  determine  $K$ . Assume that for all  $p$ ,  $T_K(\bar{D}_p; \emptyset) = 0$ . Evidently, for all couples  $(p_1, p_2)$  of arrows in  $\bar{D}$  and the corresponding configurations  $\bar{D}_{(p_1, p_2)} := \bar{D} \setminus \{p_1, p_2\}$ , we have

$$\begin{aligned} T_{K_1}(\bar{D}_{(p_1, p_2)}; \emptyset) &= \\ T_{K_2}(\bar{D}_{(p_1, p_2)}; \emptyset) & \end{aligned}$$

Therefore, we go on with considering all triples  $(p_1, p_2, p_3)$  of arrows in  $\bar{D}$ . For the configurations  $\bar{D}_{(p_1, p_2, p_3)} := \bar{D} \setminus \{p_1, p_2, p_3\}$ , we have

$$\begin{aligned} T_{K_1}(\bar{D}_{(p_1, p_2, p_3)}; \emptyset) &= \\ -T_{K_2}(\bar{D}_{(p_1, p_2, p_3)}; \emptyset) & \end{aligned}$$

If again for all triples  $(p_1, p_2, p_3)$ , one has  $T_K(\bar{D}_{(p_1, p_2, p_3)}; \emptyset) = 0$ , then we continue with 5-tuples  $(p_1, p_2, p_3, p_4, p_5)$  and so on ... At the end, we have either distinguished  $K_1$  from  $K_2$  or proven that  $T_{K_1}(c; \emptyset) = T_{K_2}(c; \emptyset)$  for any  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration  $c$  (of course,  $T_K(c; \emptyset) = 0$  for all configurations  $c$  which are not subconfigurations of  $\bar{D}$ ). One easily sees that in the latter case, the Gauss diagrams of  $K_1$  and  $K_2$  are isotopic, and hence, by Lemma 4.2,  $K_1$  and  $K_2$  are isotopic too. Theorem 2 is proven. ■

## 5 Non-invertibility of knots in $T^2 \times \mathbb{R}$

Let  $flip : T^2 \rightarrow T^2$  be the hyper-elliptic involution shown in Fig. 64.

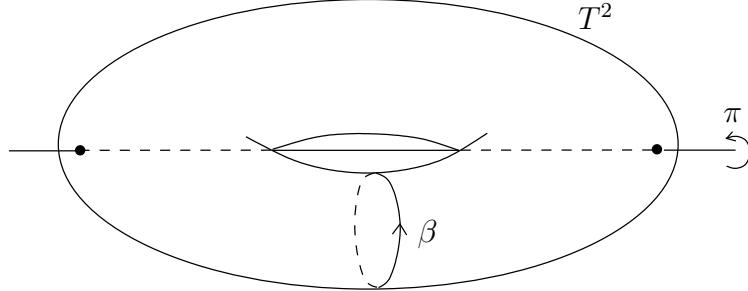


Fig. 64

The orientation preserving involution  $flip \times id$  on  $T^2 \times \mathbb{R}$  will also be called *flip* for simplicity. *flip* acts as  $-1$  on  $H_1(T^2; \mathbb{Z})$ .

Let  $K \hookrightarrow T^2 \times \mathbb{R}$  be any oriented knot, and let  $-flip(K) = flip(-K)$  be the knot obtained from  $flip(K)$  by reversing its orientation.

**Definition 5.1**  $K$  is called *invertible* if  $K$  is ambient isotopic to  $-flip(K)$  in  $T^2 \times \mathbb{R}$ . Otherwise,  $K$  is called *non-invertible*.

### Remarks.

1. We show in the next section that quantum invariants do not detect non-invertibility.
2. We show in sect. 7 that our notion of invertibility for knots in  $T^2 \times \mathbb{R}$  coincides with the usual notion of invertibility for certain links in  $S^3$ .
3. The knot  $-flip(K)$  is always homotopic to  $K$  in  $T^2 \times \mathbb{R}$

Let  $v$  be our standard vector field on  $T^2$  (see sect. 4). Let  $K \hookrightarrow T^2 \times \mathbb{R}$  be a (canonically oriented) global knot with respect to  $v$ . Let  $G$  be a quotient group of  $H_1(T^2; \mathbb{Z})/\langle [K] \rangle$ . We assume that  $K$  is a  $G$ -pure global knot. Let  $p$  be a crossing of  $K$ , and let  $p'$  be the corresponding crossing of  $-flip(K)$ .

**Lemma 5.1**  $-flip(K)$  is a  $G$ -pure global knot with respect to  $v$  too. Moreover,

1.  $w(p) = w(p')$
2.  $[K_{p'}^+] = -[K_p^+]$  in  $G$

3. Let  $D \subset \mathbb{R}^2$  be the Gauss diagram of  $K$  without writhes and homological markings. Then,  $-\text{flip}(D) \subset \mathbb{R}^2$  is obtained from  $D$  by a reflection with respect to any line in  $\mathbb{R}^2$ , followed by the reversion of the orientation of the circle.

**Proof.**  $\text{flip}(K)$  is a knot transversal to  $v$ , but with the wrong orientation. Hence,  $-\text{flip}(K)$  is a global knot.  $\text{flip} : T^2 \times \mathbb{R} \rightarrow T^2 \times \mathbb{R}$  preserves the orientation and, consequently,  $w(p) = w(p')$  for each crossing  $p$ . The involution  $\text{flip}$  maps  $K_p^+$  to  $K_{p'}^+$ . Reversing the orientation of  $\text{flip}(K)$ , the knot  $-K_{p'}^-$  for  $\text{flip}(K)$  is mapped to the knot  $K_p^+$  for  $-\text{flip}(K)$ . Thus,  $[K_{p'}^+] = [K_p^-] = [K] - [K_p^+]$  in  $H_1(T^2; \mathbb{Z})$ , and hence,  $[K_{p'}^+] = -[K_p^+]$  in  $G$ . If  $[K_p^+] \neq 0$  in  $G$ , then  $K$  is  $G$ -pure. Therefore, if  $K$  is  $G$ -pure,  $-\text{flip}(K)$  is  $G$ -pure too. Let  $D_K$  be the Gauss diagram of  $K$  without writhes and homological markings (in  $G$ ). The circle of  $D_K$  is always supposed to be embedded in the standard way in  $\mathbb{R}^2$ :  $\bigcirc D_{\text{flip}(K)}$  is exactly the same Gauss diagram (but the knots  $K$  and  $\text{flip}(K)$  are embedded in different ways), because  $\text{flip}$  preserves the orientation of the lines  $\mathbb{R}$  and, hence, preserves undercrosses and overcrosses.

Changing the orientation of  $\text{flip}(K)$  changes only the orientation of the Gauss diagram  $D_{\text{flip}(K)} = D_K$ . To obtain the standard embedding of the circle in the plane, we only need to perform a reflection with respect to a line in the plane. ■

**Definition 5.2** Let  $D$  be any  $G$ -pure configuration with markings in  $G$ . The *inverse configuration*  $\bar{D}$  is obtained from  $D$  by the successive operations:

1. a reflection with respect to a line in the plane
2. reversing the orientation of the circle
3. replacing each marking  $a \in G$  by  $-a \in G$ .

**Remark** The inverse configuration is also a  $G$ -pure configuration.

**Lemma 5.2** Let  $K \hookrightarrow T^2 \times \mathbb{R}$  be a  $G$ -pure global knot and let  $D$  be any  $G$ -pure configuration. If  $K$  is invertible, then for the  $T$ -invariants (of finite type), the following holds:

$$T_K(D; \emptyset) = T_K(\bar{D}; \emptyset)$$

**Proof.**  $K$  and  $-flip(K) = flip(-K)$  are  $G$ -pure global knots (with respect to the same  $v$ ), and they are homotopic. Lemma 5.2 follows then immediately from Theorem 1, Lemma 5.1, and the definition of the inverse configuration  $\bar{D}$ . ■

**Remark** Lemma 5.2 can be generalized in a straightforward way to the case of general  $T$ -invariants  $T_K(D; c_1(D) = c_1, \dots, c_k(D) = c_k)$  for  $G$ -pure global knots (see Def. 3.4). In particular, the *inverse class*  $\bar{c}(\bar{D})$  is defined exactly as  $c(D)$ , replacing  $D$  by  $\bar{D}$  and each configuration  $D_i$  by its inverse configuration  $\bar{D}_i$  (see Def. 3.3 and 5.2). For example, if  $K$  is invertible, then  $T_K(\emptyset; c(\emptyset)) = T_K(\emptyset; \bar{c}(\emptyset))$  for each  $G$ -pure class  $c(\emptyset)$ .

**Lemma 5.3** *Let  $K \hookrightarrow T^2 \times \mathbb{R}$  be a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot, let  $D$  be the corresponding minimal configuration, and let  $D_s \subset D$  be a subconfiguration of highest odd degree such that  $T_K(D_s; \emptyset) \neq 0$  (see sect. 4). then,  $K$  is invertible if and only if*

$$T_K(D; \emptyset) = T_K(\bar{D}; \emptyset)$$

and

$$T_K(D_s; \emptyset) = T_K(\bar{D}_s; \emptyset)$$

**Proof.** As shown in the proof of Theorem 2,  $T_K(D; \emptyset)$  and  $T_K(D_s; \emptyset)$  determine the knot  $K$ . Lemma 5.3 follows then immediately from Lemma 5.2.

■ *Example 5.1*

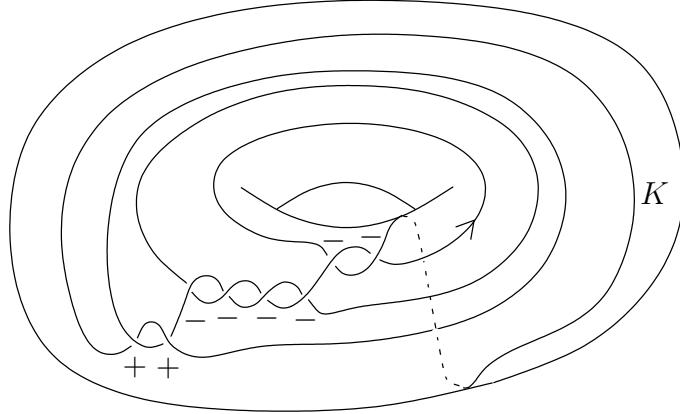
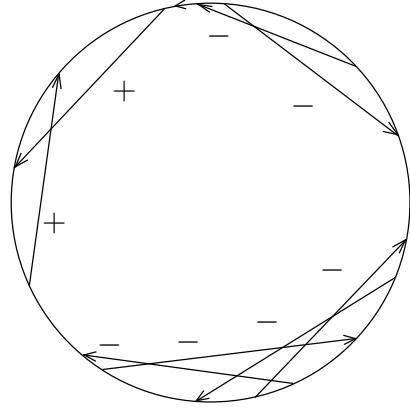


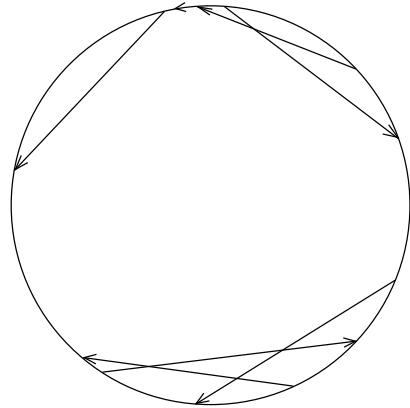
Fig. 65

The knot shown in Fig. 65 represents  $4\alpha + \beta$  in  $H_1(T^2; \mathbb{Z})$  and is a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot. Its Gauss diagram is shown in Fig. 66.



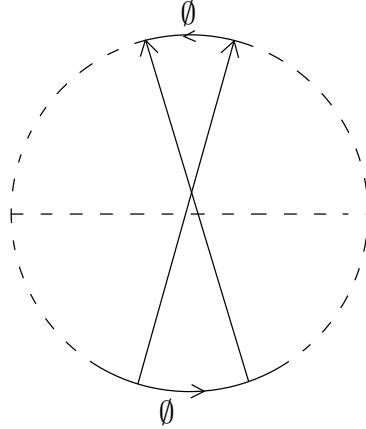
**Fig. 66**

Let  $D$  be the configuration of degree 6 shown in Fig. 67.



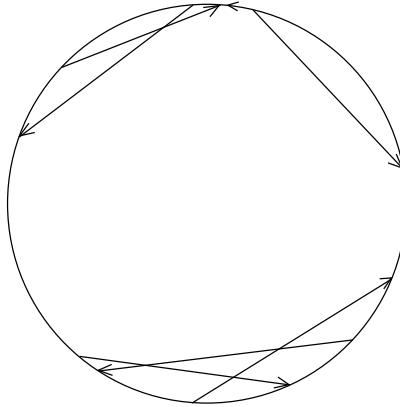
**Fig. 67**

Evidently, each configuration which does not contain a subconfiguration as depicted in Fig. 68



**Fig. 68**

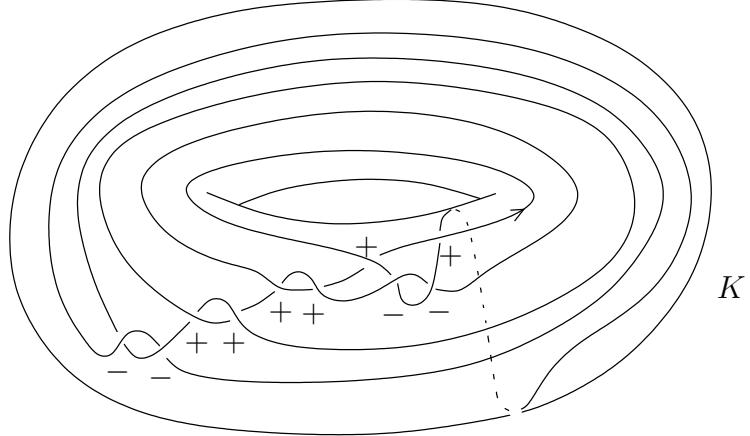
is a  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration (see Def. 3.2). Consequently,  $D$  is a  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration. Using Fig. 66 it takes some seconds to calculate  $T_K(D; \emptyset) = -1$ . The inverse configuration  $\bar{D}$  is shown in Fig. 69.



**Fig. 69**

We see immediately that  $\bar{D}$  does not appear at all in the Gauss diagram of  $K$  (the cyclic ordering of the bunches has changed). Therefore,  $T_K(\bar{D}; \emptyset) = 0$ , and the knot  $K$  is not invertible according to Lemma 5.3. *Example 5.2* This

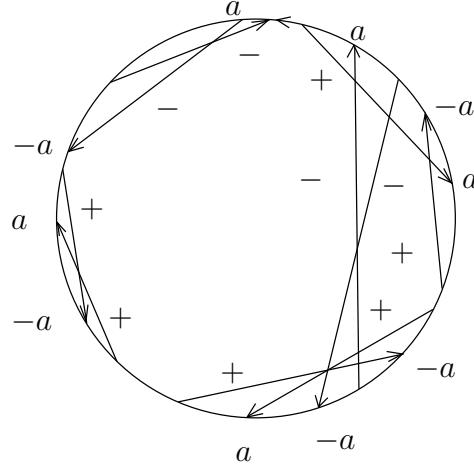
is a more complicated example.



**Fig. 70**

The knot  $K$  drawn in Fig. 70 is a global knot which represents  $5\alpha + \beta$  in  $H_1(T^2; \mathbb{Z})$ .

Let  $G := (H_1(T^2)/\langle [k] \rangle)/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} = \{0, a, -a\}$ , where the class  $a$  is represented by  $\alpha$ . The Gauss diagram of  $K$  is shown in Fig. 71.



**Fig. 71**

Hence,  $K$  is  $\mathbb{Z}/3\mathbb{Z}$ -pure. Let  $c(\emptyset)$  be the class of degree 5 shown in Fig. 72 (the weight functions are always the products of the writhes of the 5 crossings).

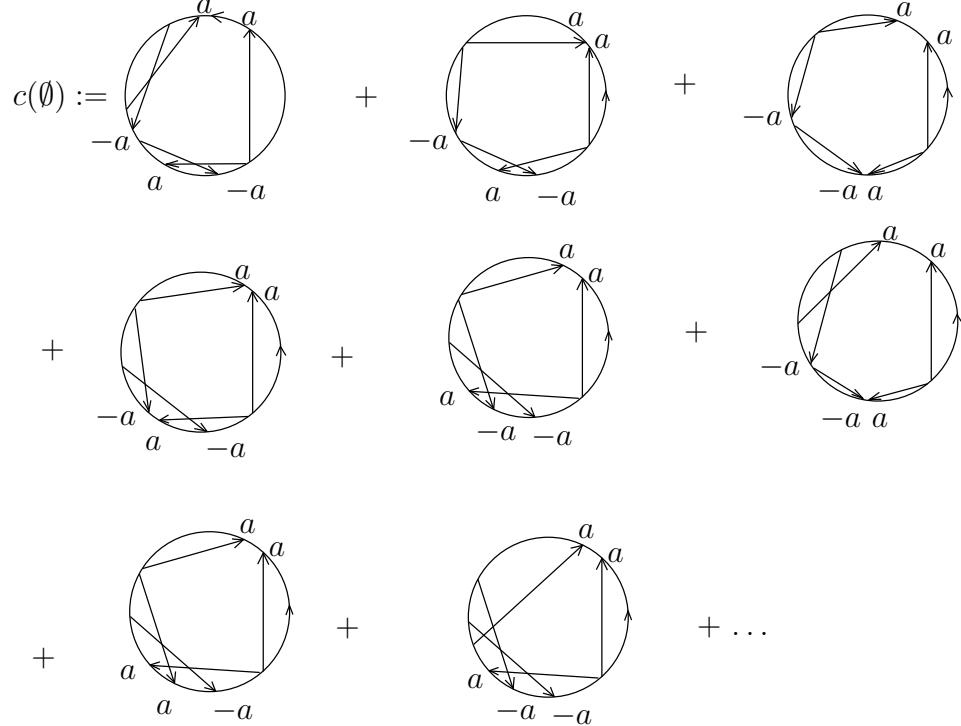


Fig. 72

The only possible strata of triple points in the discriminant for a  $\mathbb{Z}/3\mathbb{Z}$ -pure isotopy are  $a_{\text{O}(\text{O})}^{\pm}(a| -a, | -a)$  and  $a_{\text{O}(\text{O})}^{\pm}(-a|a, |a)$  (see [F], sect. 1). Therefore, the changings depicted in Fig. 73 are the only possible ones for

a couple of crossings in a  $\mathbb{Z}/3\mathbb{Z}$ -pure isotopy.

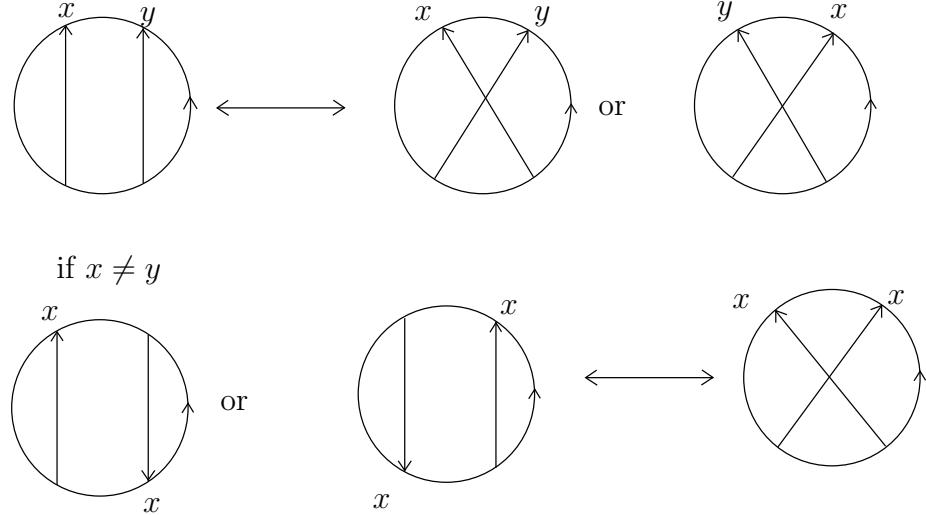


Fig. 73

Here,  $x, y \in \{a, -a\}$ .  $c(\emptyset)$  is obtained from the configuration shown in the left-hand part of Fig. 74 by applying *all* possible changings to it. We have shown some of these changings hereabove. Notice that no chord can ever get crossed with the isolated chord, because the part of the configuration shown in the right-hand part of Fig. 74 cannot change at all.

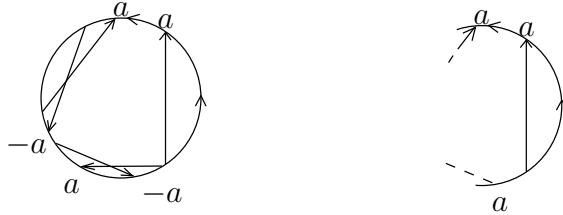


Fig. 74

Consequently,  $c(\emptyset)$  is a  $\mathbb{Z}/3\mathbb{Z}$ -pure class of degree 5 and  $T_K(\emptyset; c(\emptyset))$  is an isotopy invariant of  $K$ . Notice that  $p$  is the only arrow in the Gauss diagram of  $K$  with marking  $a$  and such that there are arrows in  $K_p^+$ . Using this fact, we easily calculate  $T_K(\emptyset; c(\emptyset)) = -1$ . For the convenience of the reader, we

give the Gauss diagram of  $flip(-K)$  in Fig. 75.

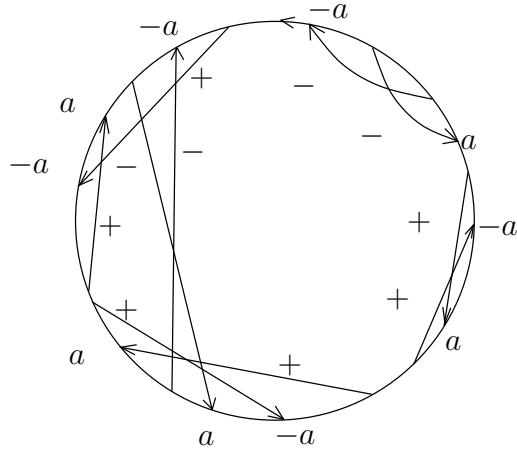


Fig. 75

Hence,  $T_{flip(-K)}(\emptyset; c(\emptyset)) = T_K(\emptyset; \bar{c}(\emptyset)) = 0$ . (Again, the cyclic order of the two couples of crossed arrows with respect to the isolated arrow  $p'$  has changed.) We have proven that  $K$  is

## 6 A remark on quantum invariants for knots in $T^2 \times \mathbb{R}$

Let  $L \hookrightarrow T^2 \times \mathbb{R}$  be any oriented link. There are generalized HOMFLY-PT and Kauffman polynomials for  $L$  (see e.g. [H-P]).

**Lemma 6.1** *The generalized HOMFLY-PT polynomials of  $L$  and of  $\text{flip}(-L)$  coincide. The generalized Kauffman polynomials of  $L$  and of  $\text{flip}(-L)$  coincide.*

**Proof.**  $L$  can be reduced to a linear combination of "initial knots" by using skein relations. These combinations are the same for  $\text{flip}(-L)$  besides the fact that each "initial knot"  $K$  has to be replaced by  $\text{flip}(-K)$ . Consequently, if we find a set of "initial knots" such that for each of them,  $K$  is isotopic to  $\text{flip}(-K)$ , then the Lemma follows. As well known, we have to choose a knot  $K$  in each free homotopy class of oriented loops in  $T^2 \times \mathbb{R}$ . Notice that " $\text{flip} \circ -$ " acts as the identity on  $\pi_1(T^2)$ .

Any primitive class in  $H_1(T^2) \cong \pi_1(T^2)$  can be represented by a torus knot which is invariant under " $\text{flip} \circ -$ ".

Let  $K \hookrightarrow T^2$ . Each class  $n[K], n \neq 0$  can be represented as the closure  $\hat{\beta}$  of the braid  $\beta = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$  in a tubular neighbourhood  $V$  of  $K \hookrightarrow T^2 \times \mathbb{R}$  which is a solid torus. (Remember that  $\sigma_i$  are the standard generators of  $B_n$ .) We easily see that  $\text{flip}(-\hat{\beta}) = \hat{\gamma}$ , where  $\gamma = \sigma_{n-1} \sigma_{n-2} \cdots \sigma_2 \sigma_1$  in the same (invariant under  $\text{flip}$ ) solid torus  $V$ . But, as well known,  $\beta$  is conjugate to  $\gamma$  in  $V$  and hence,  $\hat{\beta}$  and  $\hat{\gamma}$  are the same knot. ■

### Remarks.

1. Evidently, Lemma 6.1 is still true if one replaces  $L$  by any cable of  $L$ .
2. Lemma 6.1 implies that the above quantum invariants (and possibly all quantum invariants) can never detect non-invertibility of knots in  $T^2 \times \mathbb{R}$ .
3. It was already well known that quantum invariants never detect the non-invertibility of links in  $S^3$  (see e.g. [K]).

We have shown in sect. 5 that  $T$ -invariants detect the non-invertibility of knots in  $T^2 \times \mathbb{R}$ . Thus, these  $T$ -invariants (of degrees 5 and 6 in the examples) cannot be extracted from the HOMFLY-PT or Kauffman polynomials of the knot or any of its cables.

## 7 Non-invertibility of links in $S^3$

Our results about knots in  $T^2 \times \mathbb{R}$  can be interpreted as results about certain links in  $S^3$ . Let  $T^2 \times \mathbb{R}$  be the tubular neighbourhood of the standardly embedded torus in  $S^3$ .

Let  $T_1$  and  $T_2$  be the cores of the corresponding solid tori  $S^3 \setminus T^2$ . To each knot  $K \hookrightarrow T^2 \times \mathbb{R} \hookrightarrow S^3$ , we associate the link  $K \cup T_1 \cup T_2 \hookrightarrow S^3$ .

**Lemma 7.1** *Two knots  $K, K' \hookrightarrow T^2 \times \mathbb{R}$  are isotopic if and only if the corresponding ordered links  $K \cup T_1 \cup T_2, K' \cup T_1 \cup T_2 \hookrightarrow S^3$  are isotopic.*

**Proof.** Lemma 1.7 of [F] implies that the ordered links  $K \cup T_1 \cup T_2$  and  $K' \cup T_1 \cup T_2$  are isotopic if and only if the ordered links  $K \cup T_1$  and  $K' \cup T_2$  are isotopic in the solid torus  $S^3 \setminus T_2$ . It is also well known that each isotopy of the solid torus, which is the identity near the boundary and which maps the core of the solid torus to itself, can be isotopically deformed to an isotopy which leaves the core pointwise fixed. ■

We will use Lemma 7.1 in order to study the invertibility of the link  $K \cup T_1 \cup T_2 \hookrightarrow S^3$ . Instead of Lemma 7.1, we could use the fact that, there is only one isotopy which inverts the Hopf link  $H = T_1 \cup T_2$ , up to isotopy of isotopies. Indeed, an isotopy which inverts  $H$  inverts also the meridians and longitudes for  $T_1$  and  $T_2$ . Therefore, such an isotopy induces an orientation preserving homeomorphism of the incompressible torus  $T^2$  in  $S^3 \setminus H$ . This homeomorphism acts as  $-1$  on  $H_1(T^2; \mathbb{Z})$ . As the mapping class group of  $T^2$  is  $SL(2; \mathbb{Z})$ , this homeomorphism is isotopic to *flip*.

Thus, the non-invertibility of the link  $L$  in Fig. 1 follows from the non-invertibility of the knot  $K$  in Fig. 65. Indeed, to the knot  $K$ , we have to add the Hopf link  $T_1 \cup T_2$ . Notice that the resulting link  $L$  is naturally ordered:  $K \hookrightarrow S^3$  is not the trivial knot,  $lk(K, T_2) = 4$ ,  $lk(K, T_1) = 1$ . Hence, if  $L$  is invertible, then  $L$  is invertible as an ordered link (i.e. respecting the ordering), and one can apply Lemma 7.1. *flip*, seen as an involution on  $S^3$ , maps simultaneously  $T_1$  to  $-T_1$  and  $T_2$  to  $-T_2$ . Thus,  $L$  is isotopic to  $-L$  if and only if  $K$  is isotopic to  $flip(-K)$  in  $T^2 \times \mathbb{R}$ . But we have shown that this is not the case, using the  $T$ -invariant in Example 5.1.

## 8 *T*-invariants which are not of finite type are useful too

Let  $h = (id, -id) : T^2 \times \mathbb{R} \rightarrow T^2 \times \mathbb{R}$ , and let  $K \hookrightarrow T^2 \times \mathbb{R}$  be a global knot. Then  $h(K)$  is called the *mirror image* of  $K$  and is denoted as usually by  $K!$ . Clearly,  $K!$  is a global knot which is always homotopic to  $K$ . We give an example of a  $\mathbb{Z}/2\mathbb{Z}$ -pure global knot  $K$  which we distinguish from  $K!$ . We do this in two ways: first with a  $T$ -invariant of degree 2 but which is not of finite type, and then with a  $T$ -invariant of degree 8 which is of finite type. We prove moreover that  $K$  and  $K!$  cannot be distinguished by any Gauss diagram invariant (see [F]), or by a  $T$ -invariant of finite type of degree not bigger than 2. *Hence,  $T$ -invariants which are not of finite type are sometimes more effective than  $T$ -invariants of finite type.*

*Example 8.1*

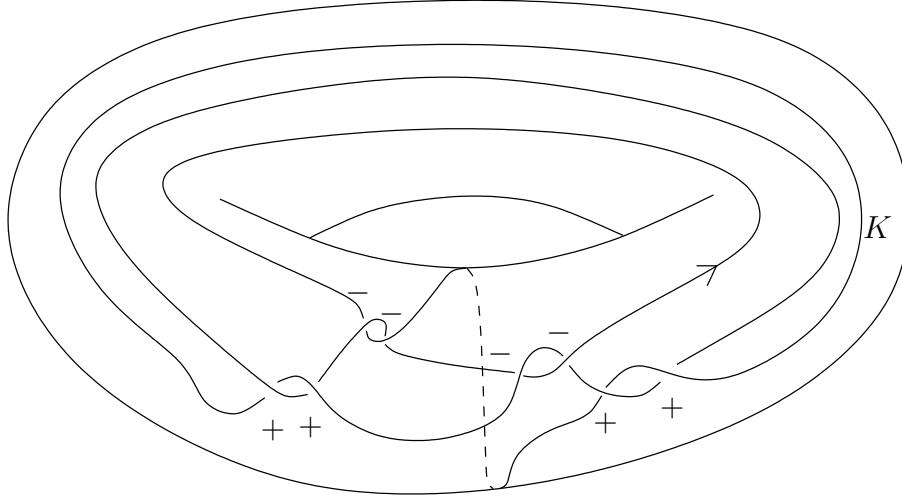


Fig. 76

The Gauss diagram of the knot in Fig. 76 is shown in Fig. 77.

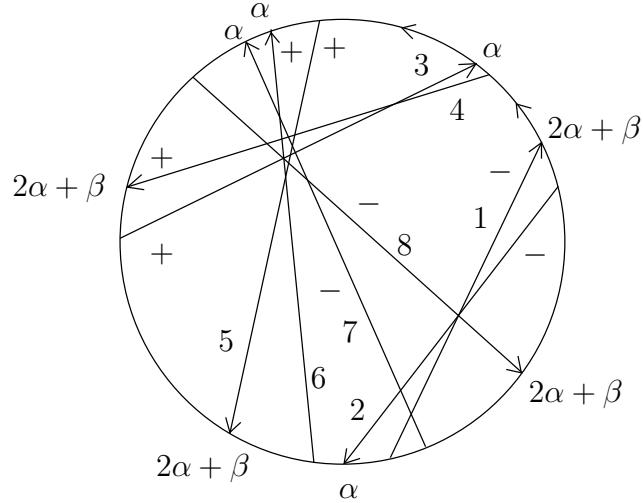


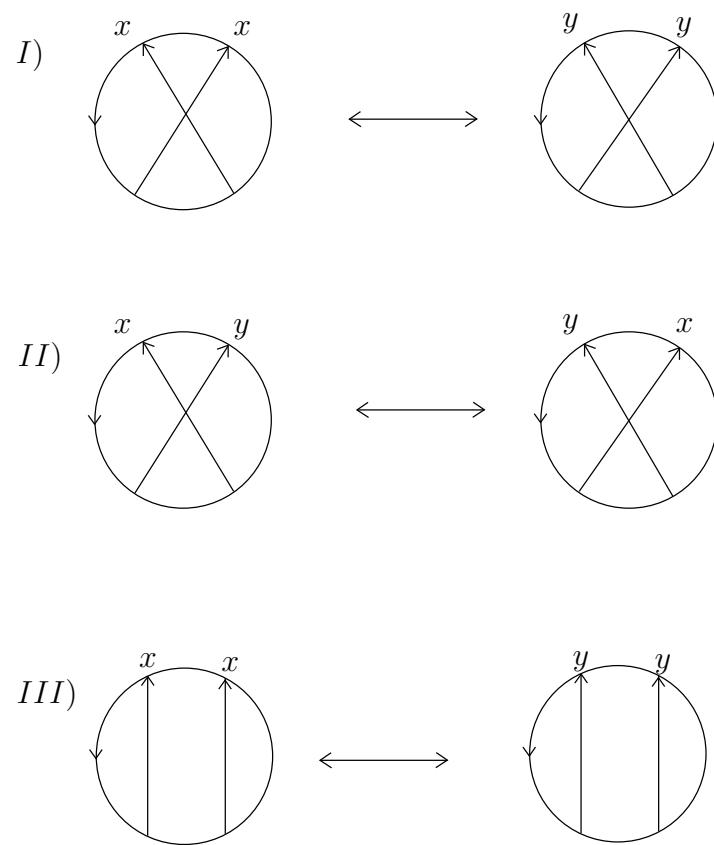
Fig. 77

For the convenience of the reader, we have affected numbers to the crossings. We see that there appear only two different homology classes as markings. In particular,  $K$  is  $\mathbb{Z}/2\mathbb{Z}$ -pure for  $G := (H_1(T^2)/\langle [K] \rangle)/2\mathbb{Z} = \{0, a\}$ . The Gauss diagram of  $K!$  is obtained from the one of  $K$  by replacing all arrows, writhes and markings by their opposites (but remember that  $a = -a$ ). We start by comparing the invariants of degree 1 (see also [F], sect. 2.2).

$$W_K(\alpha) = W_{K!}(\alpha) = W_K(2\alpha + \beta) = W_{K!}(2\alpha + \beta) = 0$$

If we see  $K$  and  $K!$  as knots in  $S^3$  using the embedding  $T^2 \times \mathbb{R} \hookrightarrow S^3$ , then we easily calculate  $v_2(K) = v_2(K!)$  for the only Vassiliev invariant of degree 2 (notice that  $K!$  is not the mirror image of  $K$  in  $S^3$  because of the two additional crossings seen in Fig. 76). All the Gauss diagram invariants and  $T$ -invariants of degree 2 which are of finite type are linear combinations of all possible configurations of degree 2 (see also [F], sect. 2.4). The weight function is always the product of the two writhes (because of the invariance under Reidemeister moves of type II). Therefore, this function

is invariant under taking the mirror image. In Fig. 78, we indicate how the configurations change by taking the mirror image.



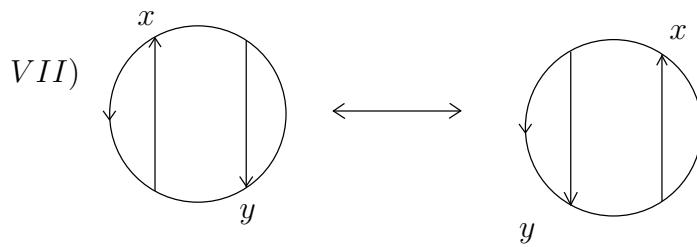
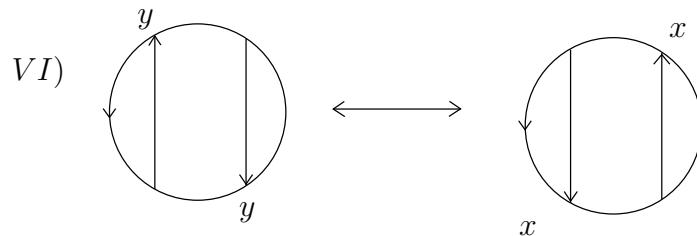
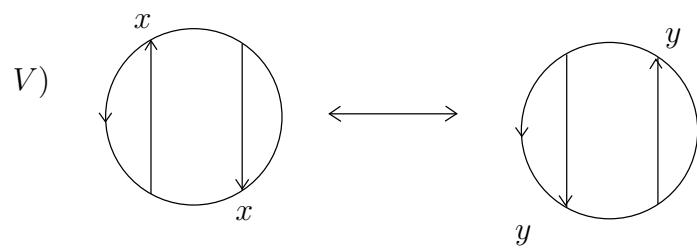
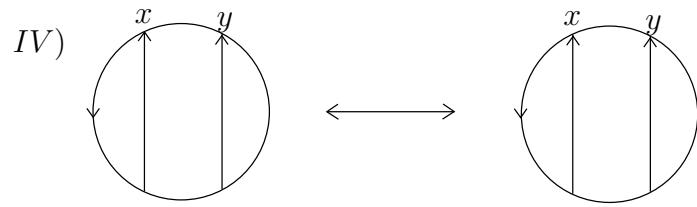


Fig. 78

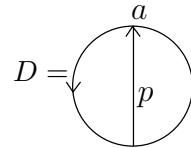
In this Figure,  $x, y \in \{\alpha, 2\alpha + \beta\}$  and  $x \neq y$ .

**Remarks.**

1.  $I$ ) can not enter in any invariant because of the invariance under Reidemeister moves of type  $II$  (see also Lemma 3.3).
2.  $IV$ ) is invariant.

Thus, if in our example, the left-hand side is equal to the right-hand side for  $II$ ,  $III$ ,  $V$ ,  $VI$ ,  $VII$ ), then *all* invariants of finite type of degree 2 coincide for  $K$  and  $K!$ . We easily calculate the values on both sides and it turns out that they coincide:  $II) = +2$ ,  $III) = 0$ ,  $V) = 0$ ,  $VI) = -2$ ,  $VII) = 0$ .

Let us consider  $T$ -invariants of infinite type (see Prop. 3.1). Let



for  $G \cong \mathbb{Z}/2\mathbb{Z} = \{0, a\}$ . We easily calculate

$$T_K(D; c_{++}(D) = +2) = -1$$

( $D$  is the crossing number 1 with the crossing  $q$  which is either the crossing number 4 or 5.)

$$T_K(D; c_{++}(D) = -1) = +2$$

( $D$  is 4 and 5,  $q$  is 1.)

$$T_K(D; c_{++}(D) = 0) = -1$$

( $D$  is 2, 3, 6, 7, 8. There are no  $q$ 's.) For all other  $c \in \mathbb{Z}$ ,  $T_K(D; c_{++}(D) = c) = 0$  We keep the numbers for the crossings of  $K!$ .

$$T_{K!}(D; c_{++}(D) = -2) = +1$$

( $D$  is 2, the  $q$ 's are 3 and 6.)

$$T_{K!}(D; c_{++}(D) = +1) = -2$$

( $D$  is 3 and 6,  $q$  is 2.)

$$T_{K!}(D; c_{++}(D) = 0) = +1$$

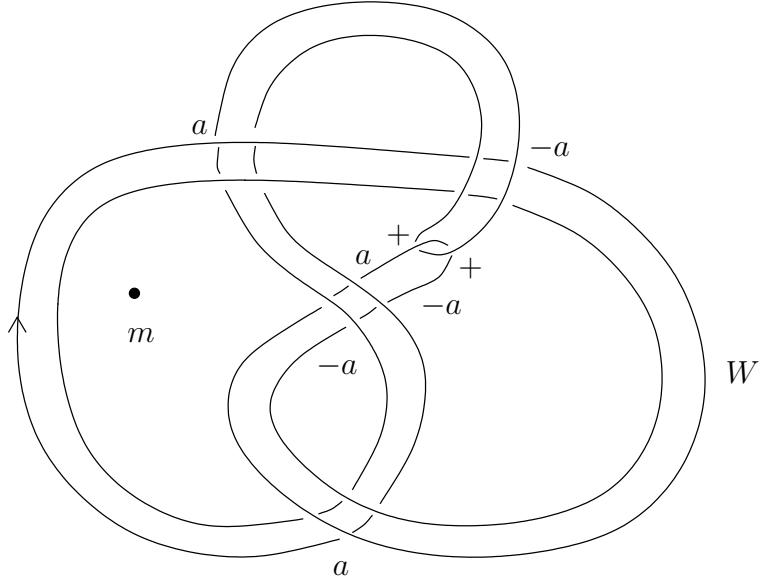
and all other  $T_{K!}(D; c_{++}(D) = c) = 0$ . Consequently,  $K$  and  $K!$  are not isotopic and we have proven it with an invariant of quadratic complexity.

The Gauss diagram of  $K$  without the writhes is a  $\mathbb{Z}/2\mathbb{Z}$ -pure configuration  $D$  of degree 8. Clearly, it is different from the corresponding configuration for  $K!$ . Thus,  $K$  and  $K!$  are also distinguished by the  $T$ -invariant of degree 8 of finite type  $T_K(D; \emptyset)$ . We do not know whether or not there are  $G$ -pure global knots which can be distinguished *only* by  $T$ -invariants which are not of finite type. But in any case, our example shows that these invariants do it sometimes in a more effective way than the invariants of finite type.

## 9 $T$ -invariants are not well defined for general knots

Let  $K \hookrightarrow S^3 = (\mathbb{R}^2 \times \mathbb{R}) \cup \{\infty\}$  be a knot and let  $m$  be a meridian of  $K$ . The meridian  $m$  is isotopic to  $(0 \times \mathbb{R}) \cup \{\infty\}$  and hence, we can consider  $K$  as a knot in  $(\mathbb{R}^2 \setminus 0) \times \mathbb{R}$ . If two knots  $K, K'$  are isotopic in  $S^3$ , then in fact, they are already isotopic in  $(\mathbb{R}^2 \setminus 0) \times \mathbb{R}$ , where  $(0 \times \mathbb{R}) \cup \{\infty\}$  is a meridian for both knots. We consider the projection  $(\mathbb{R}^2 \setminus 0) \times \mathbb{R} \rightarrow \mathbb{R}^2 \setminus 0$ . Assume that  $K$  and  $K'$  are isotopic. If they have the same writhe and the same Whitney index, then they are regularly isotopic in  $(\mathbb{R}^2 \setminus 0) \times \mathbb{R}$  (see [F], sect. 2). If we take now the same cable or satellite for two regularly isotopic knots, then the resulting knots are again (regularly) isotopic in  $(\mathbb{R}^2 \setminus 0) \times \mathbb{R}$ .

Let  $K$  be the figure-eight knot. As well known,  $K$  is isotopic to its mirror image  $K!$ . As satellite, we take the positive (untwisted) Whitehead double. Consequently, the two knots  $W$  and  $W'$  shown in Fig. 79 are isotopic in the solid torus  $S^3 \setminus m = (\mathbb{R}^2 \setminus 0) \times \mathbb{R}$ .



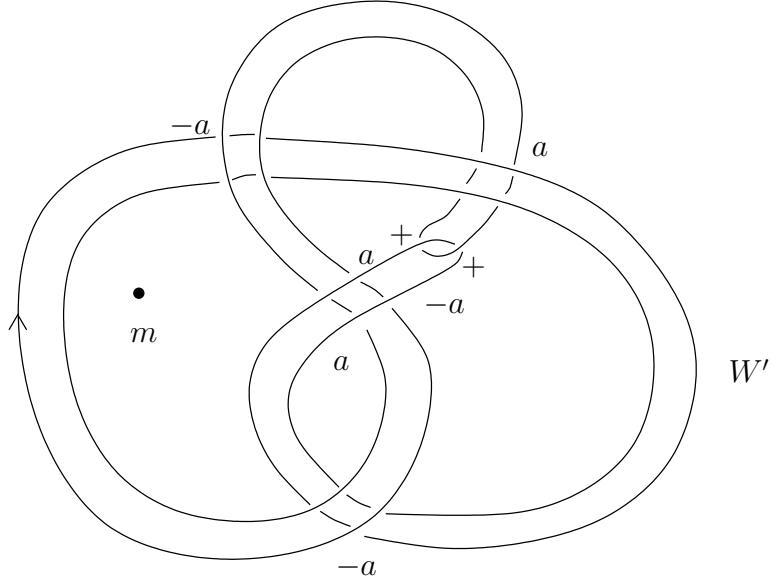
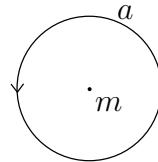


Fig. 79

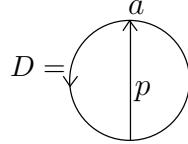
Let  $v$  be a Morse-Smale vector field on  $\mathbb{R}^2$ , which has a critical point of index 1 in  $0 = m \cap \mathbb{R}^2$ , and such that  $v$  is transversal to  $pr(W)$ . But  $pr(W) = pr(W')$  and hence,  $v$  is transversal to  $pr(W')$  too. Of course,  $W$  and  $W'$  are *not* global knots, because  $v$  has critical points of index 1 different from 0. Let "a" be the generator



of  $H_1(S^3 \setminus m; \mathbb{Z}) = H_1(\mathbb{R}^2 \setminus 0; \mathbb{Z})$ . One has  $[W] = [W'] = 0$  in  $H_1(\mathbb{R}^2 \setminus 0; \mathbb{Z})$  and we easily see that  $W$  and  $W'$  are  $\mathbb{Z}$ -pure knots (the markings are shown in Fig. 79 too). It follows from the proof of Theorem 1 that each  $T$ -invariant  $T_W$

1. is invariant for each isotopy transversal to  $v$  and which is not necessarily  $\mathbb{Z}$ -pure
2. is invariant for each  $\mathbb{Z}$ -pure isotopy which is not necessarily transversal to  $v$ .

Let



and let  $c(D)$  be any of the classes of Def. 3.7. We easily calculate  $T_W(D; c(D) = c) = T_{W'}(D; c(D) = c)$  for any  $c \in \mathbb{Z}$ . But e.g.  $T_W(D; c_{++}(D) = 0, c_{+-}(D) = -1) = +2$  and  $T_{W'}(D; c_{++}(D) = 0, c_{+-}(D) = -1) = 0$ . We have shown above that  $W$  and  $W'$  are actually isotopic in  $(\mathbb{R}^2 \setminus 0) \times \mathbb{R}$ . This example has three important consequences:

1.  $W$  and  $W'$  are transversal to  $v$  and they are isotopic. But there is no isotopy transversal to  $v$  joining them.
2.  $W$  and  $W'$  are  $\mathbb{Z}$ -pure and they are isotopic. But there is no  $\mathbb{Z}$ -pure isotopy joining them (i.e. there are cycles with marking 0 which cannot be eliminated).
3. Multi-classes contain more information than the classes taken individually.

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Unfortunately, my preprint "New invariants in knot theory" (November 1999) contains some serious errors. I apologize by the reader for this. The present preprint is the result of the correction of these errors. It will be added to the final version of the monography "Gauss diagram invariants for knots and links".

I am very grateful to Séverine for her constant support!